Introduction to Commutative Algebra

201.1.7071, Fall 2022, (D.Kerner)

Homework 9

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Questions to submit: 1.a. 1.b.ii. 1.b.iii. 1.d. 2.a. 2.c. 2.d. 3.a. 3.d. (by e-mail)

1. a. Take a short exact sequence $0 \to L \stackrel{\phi}{\to} M \stackrel{\psi}{\to} N \to 0$ and submodules $M_1, M_2 \subset M$. (Dis)Prove: if $\psi(M_1) = \psi(M_2)$ and $\phi^{-1}(M_1) = \phi^{-1}(M_2)$ then $M_1 = M_2$.
   b. (Dis)Prove: i. $S^{-1}(\cap_{j\in J} M_j) = \cap_{j\in J} S^{-1}M_j$ (for J finite)
   ii. $S^{-1}(\sum_{j\in J} M_j) = \sum_{j\in J} S^{-1}M_j$ (for any J) iii. $S^{-1}(\prod_{j\in J} M_j) = \prod_{j\in J} S^{-1}M_j$ (for any J)
   c. Take an ideal $I \subset R$. Prove: if R is a domain, then the module $I \in Mod-R$ is indecomposable. Given an example when $I \in Mod - R$ is decomposable.
   d. Given an exact sequence $L \to M \to N$ and a multiplicative set $0 \notin S \subset R$ prove: the sequence $S^{-1}L \to S^{-1}M \to S^{-1}M$ is exact. One says: "the functor $R \rightsquigarrow S^{-1}R$ is exact".
   e. Deduce: $S^{-1} \cdot M/L = S^{-1}M/S^{-1}L$ and $S^{-1}(L_1 \cap L_2) = S^{-1}L_1 \cap S^{-1}L_2$.
   f. Suppose $x \in R$ is not a zero divisor, and the image of $y \in R$ in $R/(x)$ is not a zero divisor. Verify: the sequence $0 \to R \stackrel{[-y,x]}{\to} R^2 \stackrel{[y,x]^t}{\to} R \to R/(x, y) \to 0$ is exact. (This sequence is called "the Koszul complex for the pair x, y".)
   g. Can you guess the Koszul complex for the triple x, y, z? (One assumes: $[y] \in R/(x)$ and $[z] \in R/(x, y)$ are not zero divisors.)
   h. Question 2.15 on page 48 of M.Reid's "Commutative Algebra".

2. a. Let R be Noetherian and $\phi : R \to R$ a surjective homomorphism of rings. Prove: R is an isomorphism. (Hint: consider $ker(\phi) \subseteq ker(\phi^2) \subseteq \cdots$)
   b. A topological space X is called Noetherian if any decreasing sequence of a closed sets, $X_1 \supseteq X_2 \supseteq \cdots$, stabilizes. Prove: a ring R is Noetherian iff $Spec(R)$ is Noetherian.
   c. Suppose $\Bbbk \subsetneq \bar{\Bbbk}$. Disprove: given $g, f_1..f_r \in \Bbbk[\underline{x}]$, if $g(x_o) = 0$ whenever $f_1(x_o) = 0 = \cdots = f_r(x_o)$, then $g^d \in (f_1 \dots f_r)$ for $d \gg 1$.
   d. Give examples of finitely-generated but not finitely-presented modules over the rings $\Bbbk[x_1, x_2, \dots]$, $C^\infty(\mathbb{R}^1)$, $C^\omega(\mathbb{R}^1)$.
   e. Prove: if $A \sim B$ (the left-right equivalence, see q.3 of hwk.4) then $coker(A) \cong coker(B)$.

3. (Below we assume $\Bbbk = \bar{\Bbbk}$)
   a. Describe the irreducible components of $V(J) \subset \Bbbk^3$ in the following cases:
   i. $J = (y^2 - x^4, x^2 - 2x^3 - x^2y + 2xy + y^2 - y)$ ii. $J = (xy + yz + xz, xy^2z)$
   iii. $J = ((x - z)(x - y)(x - 2z), x^2 - y^2z^3)$.
   In the cases i. and ii. find an element $f \in I(V(J)) \setminus J$.
   b. Given two varieties $V(I) \subset \Bbbk^n$ and $V(J) \subset \Bbbk^m$, prove that $V(I) \times V(J) \subset \Bbbk^n \times \Bbbk^m$ is a variety. What is its defining ideal?
   c. We have stated in the class: a (closed) subvariety $X \subset \Bbbk[\underline{x}]$ is irreducible iff $I(X) \subset \Bbbk[\underline{x}]$ is prime. Prove the part $\Leftarrow$.
   d. For any ideal $I \subset \Bbbk[\underline{x}]$ prove: $\sqrt{I} = \cap_{\mathfrak{m}\supseteq I}\mathfrak{m}$. (Recall: $\sqrt{I} = \cap_{\mathfrak{p}\supseteq I}\mathfrak{p}$.)
   e. Deduce the weak Nullstellensatz for the rings $\Bbbk[\underline{x}]/I$, $S^{-1}\Bbbk[\underline{x}]$ from that for $\Bbbk[\underline{x}]$. (We did the case $S^{-1}\Bbbk[\underline{x}]$ in the class.)