Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 10. Submission date: 12.06.2023

Questions to submit: 1.c. 2.a. 2.b. 2.e. 3.b. 3.c. 4.b. 4.c.

Either typed or in readable handwriting and scanned in readable resolution.

1. Let $\mathbb{X}(t)$ be a fundamental matrix of solutions of the ODE $\underline{x}' = A(t)\underline{x}$. Prove:

a. Any two fundamental matrices are related by $\tilde{\mathbb{X}}(t) = \mathbb{X}(t) \cdot U$ for $U \in GL(n, \mathbb{R})$ (a constant matrix).

b. $\mathbb{X}(t)$ is non-degenerate for each t and satisfies: $\mathbb{X}'(t) = A(t) \cdot \mathbb{X}(t)$.

c. Given the initial condition $\underline{x}(t_0) = \underline{x}_0$, the solution is: $\mathbb{X}(t) \cdot \mathbb{X}^{-1}(t_0) \cdot \underline{x}_0$.

2. a. Given two functions $x_1(t), x_2(t) \in C^1(a, b)$ (not necessarily solutions of some ODE), suppose $W(x_1(t), x_2(t)) = 0$ on (a, b). Does this imply the $\mathbb{R}$-linear dependence of $x_1(t)$, $x_2(t)$? (Hint at the end of page)

b. Prove: if $W(x_1(t), \dots, x_n(t)) = 0$ on (a, b) for some analytic functions, then these functions are $\mathbb{R}$-linearly dependent on (a, b).

c. Suppose a solution $x(t)$ of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} \cdots + a_0(t)x = 0$, with $a_j \in C^0$ has infinitely many zeros on a compact interval. Prove: $x(t) = 0$ on this interval.
Can the compactness be weakened to boundedness here?

d. Prove: the function $sin(t^p)$, $p \in \mathbb{N}$, cannot be a solution of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ with C^0-coefficients, for $n < p$.

e. Prove: the function $e^{-\frac{1}{t^2}}$, extended to $(-1, 1)$, cannot be a solution of equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ with C^0-coefficients, for any n.

3. a. Let $\mathbb{X}(t) := [\underline{x}_1(t), \dots, \underline{x}_n(t)] \in Mat_{n\times n}(C^1(a, b))$, here the columns are some solutions of $\underline{x}' = A(t) \cdot \underline{x}$. Prove: $det[\mathbb{X}(t)] \not\equiv 0$ iff $\mathbb{X}(t)$ is non-degenerate for all $t \in (a, b)$.

b. Find a system $\underline{x}' = A(t) \cdot \underline{x}$ whose solutions are $\underline{x}_1(t) = [e^t cos(t), e^t sin(t)]$ and $\underline{x}_2(t) = [-sin(t), cos(t)]$.

c. Prove: if $\lim_{t\to\infty} \int^t trace[A(s)]ds = \infty$, then at least one solution of $\underline{x}' = A(t)\underline{x}$ is unbounded.
Show by an example that the conclusion "$\|\underline{x}(t)\| \to \infty$ for at least one solutions" fails.

d. Prove: the rescaling $\underline{x} \to e^{-\int^t \frac{trace[A(s)]}{n} ds}\underline{x}$ transforms $\underline{x}' = A(t) \cdot \underline{x}$ into the system $\underline{x}' = \tilde{A}(t) \cdot \underline{x}$ with $trace[\tilde{A}(t)] = 0$.

4. a. Prove: there exists a fundamental matrix $\mathbb{X}(t)$ of the equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots = 0$ satisfying $\mathbb{X}(t_0) = \mathbb{1}$.

b. Verify: the functions $sin(t^2)$, $cos(t^2)$ are (linearly independent) solutions of $tx'' - x' + 4t^3x = 0$, but the Wronskian of these functions vanishes at a point. Any contradiction to **3.a**?

c. Write the general solution of $tx'' + 2x' - tx = 0$. (Hint: one solution is $x(t) = \frac{e^t}{t}$.)

d. Find a linear ODE whose space of solutions is spanned by $sin\frac{1}{t}$, $cos\frac{1}{t}$.

e. Prove: the rescaling $x \to e^{-\int^t \frac{a_{n-1}(s)}{n} ds}x$ transforms the equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$ into an equation with $\tilde{a}_{n-1}(t) = 0$.

Hint: x^2, $x \cdot |x|$.