Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 11. Submission date: 17.06.2023

Questions to submit: 1.b. 1.c. 2.a. 2.c. 2.d. 3.a. 4.b. 4.d.

Either typed or in readable handwriting and scanned in readable resolution.

1. a. Suppose $A(t) = \begin{bmatrix} sin(t) & arctan(cos(2t)) \\ cos(sin(2t)) & \frac{1}{10} - cos(t) \end{bmatrix}$. Prove: the system $\underline{x}' = A(t) \cdot \underline{x}$ has at least one unbounded solution.
   b. For any solution of $\underline{x}' = A(t)\underline{x}$ prove the Grönwall bound: $\|\underline{x}(t)\| \leq \|x(t_0)\| \cdot e^{\int_{t_0}^{t} ||A(s)||_{op} ds}$.
   c. Obtain a similar bound for any solution of $\underline{x}' = A(t)\underline{x} + \underline{b}(t)$.

2. Consider the equation $D_n(x) = g(t)$, where $D_n = \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0$, with $a_i \in \mathbb{R}$.
   a. Write the general solution of $x^{(4)} + 4x = \sum b_j e^{\omega_j t}$, here $\omega_j \in \mathbb{C}$, with $\omega_j = 0$ or $\omega_j^3 = -4$.
   b. Suppose $\mu \in \mathbb{C}$ is not a root of the characteristic polynomial of D_n. Prove: the equation $D_n(x) = t^k \cdot e^{\mu t}$, with $k \in \mathbb{N}$, has a solution of the form $g_k(t) \cdot e^{\mu t}$ for a polynomial $g_k(t) \in \mathbb{C}[t]_{\leq k}$ of degree k. (Hint. It is enough to show: the operator $D_n \circlearrowright \mathbb{C}[t]_{\leq k} \cdot e^{\mu t}$ acts surjectively. And for this it is enough to verify: D_n acts injectively.)
   c. Suppose $\mu \in \mathbb{C}$ is a root of the characteristic polynomial of D_n, of multiplicity p. Prove: the equation $D_n(x) = t^k \cdot e^{\mu t}$ has a solution of the form $t^p \cdot g(t) \cdot e^{\mu t}$ for a polynomial $g_k(t) \in \mathbb{C}[t]_{\leq k}$ of degree k. Wiki: "Resonance".
   d. Write the general solution of $x^{(4)} + 4x = b \cdot t \cdot e^{\mu t}$. (Here $b, \mu \neq 0$ are parameters.)
   e. Consider the equation $D_n x = p(t) \cdot e^{\mu t}$, here $p(t) \in \mathbb{C}[t]$. What is the necessary and sufficient condition to ensure that the equation has a periodic solution? A bounded solution?

3. a. Suppose $e^t, sin(t), t^{17}$ are solutions of a linear non-homogeneous equation of 2'nd order. Write the general solution. Find the solution satisfying: $x(0) = a$, $x'(0) = b$.
   b. Suppose the functions $x_1(t) \dots x_n(t)$ are $\mathbb{R}$-linearly independent. Take the differential equation $D_n x = 0$ whose space of solutions is $Span_{\mathbb{R}}(x_1(t), \dots, x_n(t))$. Prove: if all $\{x_i(t)\}$ are T-periodic then so are the coefficients of the operator D_n.

4. a. Suppose $x(t)$ is a solution of $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$. We have seen how one can use $x(t)$ to pass to an equation $y^{(n-1)} + \tilde{a}_{n-2}(t)y^{(n-2)} + \cdots + \tilde{a}_0(t)y = 0$. Prove: $x(t)$ together with the solutions of this last equations provide the complete system of solutions of the initial equations. In particular, given the independent solutions $y_1(t), \dots, y_{n-1}(t)$ verify: the functions $x(t), x(t) \cdot \int^t y_1(s)ds, \dots, x(t) \cdot \int^t y_{n-1}(s)ds$ are $\mathbb{R}$-linearly independent.
   b. Find the general solution of $tx'' - (t+n)x' + nx = 0$, for $n \in \mathbb{N}$, given a solution e^t.
   c. Find the general solution of $(t^2 - 1)x'' + 4tx' + 2x = 6t$, given the particular solutions $x_1(t) = t$, $x_2(t) = \frac{t^2+t+1}{t+1}$.
   d. Find the general solution of Bessel's equation $t^2x'' + tx' + (t^2 - \frac{1}{4})x = 3t^{\frac{3}{2}} sin(t)$, $t > 0$, given a particular solution of the corresponding homogeneous equation, $x(t) = \frac{sin(t)}{\sqrt{t}}$.