Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 12. Submission date: 25.06.2023

Questions to submit: 1.c. 1.d. 1.e. 2.c. 2.d. 2.e. 3.b.

Either typed or in readable handwriting and scanned in readable resolution.

1. a. Compute the monodromy matrix, e^{RT}, for the system $\underline{x}' = A \cdot \underline{x}$, with constant A.
   b. Recall, the fundamental matrix $\mathbb{X}(t)$ of the (periodic) system $\underline{x}' = A(t) \cdot \underline{x}$ is non-unique. Prove: the monodromy matrix is well defined up to conjugation and does not depend on the choice of t_o.
   c. Suppose the coefficients of $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots = 0$ are periodic. Prove: for any basis of solutions $\underline{x}(t) := (x_1(t), \dots, x_n(t))$ one can present $\underline{x}(t) = \underline{y}(t) \cdot e^{Rt}$, where $\underline{y}$ is a row of periodic functions, while $R \in Mat_{n\times n}(\mathbb{C})$.
   d. Let $x_1(t), x_2(t)$ be solutions of the equation $x'' + a(t)x' + b(t) = 0$, where $a(t), b(t) \in C^0(\mathbb{R})$ are periodic with period T. Take the corresponding fundamental matrix $\mathbb{X}(t)$, we assume $\mathbb{X}(0) = \mathbb{1}$. Let e^{RT} be the corresponding monodromy matrix. Prove: its eigenvalues are the roots of polynomial $z^2 - (x_1(T) + x_2'(T))z + e^{-\int_0^T a(t)dt} = 0$.

2. Suppose $0 \not\equiv x(t) \in C^2[t_1, t_2]$ is a solution of the ODE $x'' + a(t)x = 0$.
   a. Suppose $a(t) \leq 0$ on $[t_1, t_2]$. Prove: $x(t)$ can have at most one zero on (t_1, t_2).
   b. Suppose $a(t) \leq k^2$ on $[t_1, t_1 + \frac{\pi}{k}]$. Prove: $x(t)$ can have at most one zero on $(t_1, t_1 + \frac{\pi}{k})$.
   c. (Sturm separation theorem) Let $x_1(t)$, $x_2(t)$ be linearly independent solutions of the equation $x'' + a_1(t)x' + a_0 x = 0$. Prove: between any two consecutive zeros of $x_1(t)$ lies exactly one zero of $x_2(t)$.
   d. Prove: if $0 < m \leq a(t) \leq M$ on (t_0, ∞), then the distance between any two consecutive zeros of $x(t)$ is $\frac{\pi}{\sqrt{M}} \leq dist \leq \frac{\pi}{\sqrt{m}}$.
   e. (Kneser's theorem) If $a(t) > \frac{1+\epsilon}{4t^2}$ for some $\epsilon > 0$, then $x(t)$ has infinite number of zeros. If $a(t) < \frac{1}{4t^2}$ then $x(t)$ has a finite number of zeros.
   (Hint: compare to the Euler equation $x'' + \frac{1}{4t^2}x = 0$.)

3. The Bessel function $J_n(t)$ is defined as the solution of the equation $x'' + (1 + \frac{1/4 - n^2}{t^2})x = 0$.
   a. Read Wiki-page about the applications of Bessel functions.
   b. Prove: $J_n(t)$ has infinite number of zeros, say $\{t_k\}$, and $\lim_{k\to\infty} |t_{k+1} - t_k| = \pi$.
   c. When does one have $|t_{k+1} - t_k| > \pi$? When does one have $|t_{k+1} - t_k| < \pi$?
   d. Prove: if $n > m$ then between every two zeros or J_m lies a zero of J_n.