Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 13. Submission date: 1.07.2023

Questions to submit: 1.a.ii. 1.c. 2.a. 2.d. 3.a. 3.d. 4.b. 4.c.

Either typed or in readable handwriting and scanned in readable resolution.

1. a. Study the stability of the equilibria points for the systems:
   i. $x' = e^{x+2y} - cos(3x)$, $y' = \sqrt{4+8x} - 2e^y$. ii. $x' = x^2 + y^2 - 1$, $y' = 2xy$.
   b. Prove that the following conditions are equivalent:
   i. Any solution of $\underline{x}' = A(t)\underline{x} + \underline{b}(t)$ is stable.
   ii. At least one solution of $\underline{x}' = A(t)\underline{x}$ is stable.
   iii. all the solutions of $\underline{x}' = A(t)\underline{x}$ are bounded for $t \to \infty$.
   (In particular: either all solutions are stable or all are unstable.)
   c. (Dis)Prove: if all the solutions of $x' = f(x)$ are bounded then the equilibria points are stable.

2. a. Given the system $\underline{x}' = A(t)\underline{x} + \underline{b}(t)$, denote by $\lambda_{max}(t)$ the largest eigenvalue of the matrix $\frac{A(t)+A^T(t)}{2}$ (for each time moment). Prove:
   i. If $-\infty \le \int_{t_o}^{\infty} \lambda_{max}(s)ds < \infty$ then any solution is stable.
   ii. If $\int_{t_o}^{\infty} \lambda_{max}(s)ds = -\infty$ then any solution is exponentially-stable.
   b. Consider the equation $x^{(n)} + a_{n-1}(x, x', \dots, x^{(n-1)}) \cdot x^{(n-1)} + \dots + a_0(x, x', \dots, x^{(n-1)}) \cdot x = 0$, for some continuous functions $\{a_i(..)\}$. Prove: if $Re(\lambda) < 0$ for all the roots of the polynomial $\lambda^n + a_{n-1}(o)\lambda^{n-1} + \dots + a_0(o)$ then the zero solution is stable.
   c. Consider the system $\underline{x}' = A(t) \cdot \underline{x}$, where $A(t) \in Mat_{n\times n}(C^0(0,\infty))$. Suppose any solution satisfies: $\lim_{t\to\infty} \|\underline{x}(t)\| = 0$. Prove: $\int_{t_o}^{\infty} trace[A(s)]ds = -\infty$.
   d. Suppose $x = 0$ is an asymptotically stable solution for the system $\underline{x}' = A\underline{x}$, where A is $\mathbb{C}$-diagonalizable. Suppose $\int_0^{\infty} \|B(s)\|_{op} ds < \infty$. Prove: any solution of the system $\underline{x}' = (A + B(t))\underline{x}$ is asymptotically stable.

3. a. Let $\mathbb{X}(t), \tilde{\mathbb{X}}(t)$ be two fundamental matrices of the system $\underline{x}' = A(t)\underline{x}$. Suppose $\mathbb{X}(t_1) = 1\!\!1$ and $\tilde{\mathbb{X}}(t_2) = 1\!\!1$. Prove: $\mathbb{X}(t_2) \cdot \tilde{\mathbb{X}}(t_1) = 1\!\!1$.
   b. Let $\mathbb{X}_A(t)$ be a fundamental matrix for $\underline{x}' = A(t) \cdot \underline{x}$, let $\mathbb{X}_B(t)$ be a fundamental matrix for $\underline{x}' = B(t) \cdot \underline{x}$. Prove: if $\mathbb{X}_A(t)B(t) = B(t)\mathbb{X}_A(t)$ then $\mathbb{X}_A(t)\mathbb{X}_B(t)$ is a fundamental matrix of $\underline{x}' = (A(t) + B(t))\underline{x}$.
   c. Prove: λ is an eigenvalue of the monodromy matrix e^{RT} iff there exists a solution $\underline{x}(t)$ satisfying $\underline{x}(t+T) = \lambda \cdot \underline{x}(t)$.
   d. Let $\underline{x}' = A(t)\underline{x}$ where $A(t)$ is periodic with period T. Take the fundamental matrix $\mathbb{X}(t)$ satisfying: $\mathbb{X}(0) = 1\!\!1$. Prove: $\mathbb{X}(d \cdot T) = \mathbb{X}(T)^d$.

4. a. Let $A(t)$ be a periodic matrix. Prove: there exists a constant matrix $C \in Mat_{n\times n}(\mathbb{C})$ such that the system $\underline{x}' = (A(t) - C)\underline{x}$ has a basis of periodic solutions.
   b. Suppose two solutions, $x(t), \tilde{x}(t)$ of $D_n x = 0$ satisfy: $\tilde{x}^{(i)}(t_o + T) = x^{(i)}(t_o)$ for all $i = 0, ., n-1$. Prove: $x(t) = \tilde{x}(t+T)$.
   c. Suppose the operator D_2 has periodic coefficients. Suppose a non-trivial solution of $D_2 x = 0$ has at least two zeros. Prove: any solution has infinity of zeros.