

Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 2. Submission date: 01.04.2023

Questions to submit (by mail): 1.ii-iv. 2a. 2d. 2g. 3c. 4c. 4d.
Either in Latex or in readable handwriting and in readable resolution.



- In the following cases draw the integral curves. When possible write down the general solution. In each case extend the solution to the maximal interval. Does a solution explode at a finite time? Which solutions are monotonic/bounded/periodic? When you have a constant solution, is this an (un)stable equilibrium point? For which initial conditions does the solution of the corresponding IVP exist/is unique? Check the C^r -properties for $1 \leq r \leq \infty, \omega$.
 - $x' = \frac{t+|t|}{x+|x|}$
 - $x' = e^x \cdot \sin(x)$
 - $x' = \frac{1}{\sqrt[3]{\sin(x)}}$
 - $x' = e^{\frac{1}{x}}$
 - $x' = \frac{\sin(x)}{\sin(t)}$
 - $x' = \sqrt{|x|} \cdot \sqrt[3]{\sin(x)}$
- Consider the equation $x' = f(x)$, $f \in C^0(a, b)$, $(a, b) \subseteq \mathbb{R}^1$, and its solution $x(t)$.
 - Show that there is a continuum of global solutions, $x(t) \in C^1(\mathbb{R})$, to the IVP $x' = \sqrt{|x|}$, $x(t_0) = -1$. Check that they are all C^1 but not C^2 .
Find the largest interval around t_0 on which the solution to this IVP is unique.
 - Give an example of non-differentiable f such that all the solutions of $x' = f(x)$ are real-analytic. (Hint at the end of page)
 - Suppose the function $f \in C^0(\mathbb{R})$ is periodic. Prove: $x(t+T) = x(t) + p$ for some constants T, p and any t . (These functions are called “arithmetic quasiperiodic”.)
 - Suppose $f(\pi) = 0$ and $\int_{\pi}^{\pi+\epsilon} \frac{dx}{f(x)} = \infty$. Find all the solutions that satisfy $x(\sqrt{2}) = \pi$.
 - Suppose $f \in C^0(-\epsilon, \epsilon)$ and $f(x) = O(x \cdot \ln|x|)$. (Does this condition imply that f is locally Lipschitz?) Prove: the solution of the IVP $x' = f(x)$, $x(0) = 0$ is locally unique.
 - Prove: any solution $x(t)$ is (weakly) monotonic.
 - Suppose f is locally Lipschitz at each point, and the set of zeros of f is unbounded in both directions. Prove: every local solution extends to the unique global solution, $x(t) \in C^1(a, b)$.
- Suppose $x(t)$ is a solution of the equation $x' = f(t, x)$, here $f \in C^0(\mathcal{U})$ for an open $\mathcal{U} \subseteq \mathbb{R}^2$.
 - Solve the equation $x' = -x + g(t) + g'(t)$, here $g \in C^1(a, b)$.
 - Suppose $f(t, -x) = -f(t, x)$ for all $(t, x) \in \mathbb{R}^2$. Prove: $-x(t)$ is a solution as well.
In the case of uniqueness conclude: either $x(t) \equiv 0$ or $x(t)$ has no zeros.
 - Suppose $f \in C^1(\mathcal{U})$ and $x(t), y(t) \in C^1(a, b)$ are two solutions satisfying $x(t_0) < y(t_0)$.
Prove: $x(t) < y(t)$ for any $t \in (a, b)$. Is the assumption $f \in C^1(\mathcal{U})$ necessary here?
- Consider the equation $x' \cdot \sin(t) = x \cdot r \cdot \cos(t)$ for $r \in \mathbb{N}$, with $r \geq 2$.
 - Write down the general local solution near $t_0 \in \mathbb{R} \setminus \pi\mathbb{Z}$. Which initial conditions, (t_0, x_0) , are allowed?
 - Prove: every local solution at $t_0 \in \mathbb{R} \setminus \pi\mathbb{Z}$ extends (uniquely) to a global solution $x(t) \in C^\infty(\mathbb{R})$.
 - Prove: for any number sequence $\{x_k\}_k$ there exists a C^{r-1} -solution satisfying: $\{x(\frac{\pi}{13} + \pi k) = x_k\}_k$. (Any contradiction to the uniqueness theorem?)
 - Prove: the set of C^{r-1} -solutions is a vector space of uncountable dimension.
 - Prove: the set of C^r -solutions is a vector space of dimension 1.
Prove: any C^r -solution is in fact a C^∞ -solution.
- As a motivation for “Vector fields integration” (and systems of ODE’s), read wiki “Eversion of spheres” and watch youtube “Turning spheres inside-out”.

Hint: $|x| = x$