Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 3. Submission date: 13.04.2023

Questions to submit: 1.b. 2.i. 2.iii. 2.v. 4.c. 5.a. 6.b.
Homeworks must be either typed (e.g. in Latex) or written
in readable handwriting and scanned in readable resolution.

1. The Bernoulli equation, $x' = g(t)x + h(t)x^{n+1}$, $g, h \in C^0(a,b)$, $n \neq 0, -1$, is important for fluid transition between two reservoirs. The standard approach is to convert it to a linear equation, by substition $x = z^k$ near a point $x_0 \neq 0$.
a. Find the needed k. (Address both cases $x_0 > 0$, $x_0 < 0$.)
b. Solve the IVP: $x' = x \cdot tan(t) + x^4 \cdot cos(t)$, $x(\pi) = -1$.

2. In the following cases write the general solution (at least in the form $F(x,t) = const$). Draw the integral curves (you can use any software). For which initial conditions the local solution exists/is unique? What are the singular points of these curves (the points with $grad(F) = \vec{0}$)?
i. $x' = \frac{2t+3t^2x}{3x^2-t^3}$ ii. $x' = \frac{2x}{3t} + \frac{2t}{x}$ iii. $3t^2x + x^2 + 2t^3x' + 3tx \cdot x' = 0$
iv. $x' = x(1 + xe^t)$ v. $tx' - 2t^2\sqrt{x} = 4x$

3. Take two linear functions $l_i(t,x) = a_i t + b_i x - c_i$, $i = 1, 2$. Consider the equation $x' = \frac{l_2(t,x)}{l_1(t,x)}$.
a. Suppose the lines $\{l_i(t,x) = 0\} \subset \mathbb{R}^2$ are not parallel. Shift the coordinates $t \to t - c_t$, $x \to x - c_x$ to move the line intersection to the origin. In the new coordinates the equation becomes homogeneous. Write the details. Solve the IVP: $x' = \frac{4x-2t-6}{x+t-3}$, $x(2) = 2$.
b. If the two lines are parallel (but do not coincide) then change the coordinates $x \to x - c \cdot t$ to get an autonomous equation. Write the details.
Write the general solution of the equation $x' = \frac{t-x-1}{t-x-2}$.

4. For a differentiable function $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{F} \mathbb{R}$ define the total differential $dF := \sum_j \partial_j F \cdot dx_j$.
a. Is $xdy - ydx$ the total differential of a C^2-function?
b. Prove: if $\sum a_j \cdot dx_j$ is the total differential of a C^2 function then $\partial_j a_i = \partial_i a_j$ for all i, j.
c. Prove: the differential forms $\frac{xdx+ydy}{(x^2+y^2)^p}$, $\frac{ydx-xdy}{(x^2+y^2)}$ are locally exact at each point of $\mathbb{R}^2 \setminus \{0,0\}$. Let $\mathcal{U}$ be one of: $\mathbb{R}^2 \setminus \{0,0\}$, $\mathbb{R}^1_{x>0} \times \mathbb{R}^1_y$, $\mathbb{R}^2 \setminus \{y = 0, x \leq 0\}$. In which cases are these forms exact on $\mathcal{U}$?

5. Pass from the (x,t)-coordinates to the polar coordinates (r, ϕ). Accordingly transform the equation $x' = f(\frac{x}{t})$ into $\frac{dr}{d\phi} = r \cdot \Phi(\phi)$. Here the function Φ is 2π-periodic and possibly not defined on some (discrete) set of points. Define $p := \int_0^{2\pi} \Phi(\phi) d\phi$, suppose this integral converges.
a. Describe the integral curves in the cases $p > 0$, $p = 0$, $p < 0$. Prove: any local solution extends to a global one, $\mathbb{R}^1 \xrightarrow{r(\phi)} \mathbb{R}^1_{\geq 0}$. Conclude: in (x,t) coordinates no local solution extends to a global $\mathbb{R}^1 \xrightarrow{x(t)} \mathbb{R}^1$, and for some initial conditions (which?) there is no existence/uniqueness.
b. As an explicit example analyze the solutions of $x' = \frac{C \cdot x - t}{x + C \cdot t}$, for all constants $C \in \mathbb{R}$.

6. Consider the equation $f \cdot dx + g \cdot dy = 0$. Suppose $g(x_0, y_0) \neq 0$.
a. Prove: there exists a local integrating factor of the form $\lambda(x)$ iff $\frac{f-g}{g}$ is locally independent of y. In this case $\lambda = C \cdot e^{\int^x (\frac{f-g}{g}) ds}$. (We have proved this in the class.)
b. Obtain similar results for the existence of integrating factors in the forms $\lambda(x-y)$, $\lambda(x^2+y^2)$.
c. Solve the equations: i. $(t^2 - x^2 + 1) + (t^2 - x^2 - 1)x' = 0$,   ii. $ydx - (x^2 + y^2 + x)dy = 0$.