Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 4. Submission date: 25.04.2023

Questions to submit: 1.b. 1.d. 2.b. 2.d. 3.b. 3.c.ii. 4.c. 4.d.
Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. a. Prove: the solutions of $x' = e^{x^2} - t$ have no local minima. ($\exists$ at least two different approaches.)

b. Prove: every local solution of $x' = sin^2(t) \cdot e^{t \cdot cos(x)}$ extends (uniquely) to $x(t) \in C^\omega(\mathbb{R})$, this global solution has infinite number of critical points, and all the critical points are flexes (i.e. neither maxima nor minima).

c. Prove: the local solution of $x' = \frac{(x-1)sin(t \cdot x)}{t^2+x^2+1}$, $x(0) = \frac{1}{2}$ extends (uniquely) to the global solution, $x(t) \in C^\omega(\mathbb{R})$. Moreover it satisfies: $0 < x(t) < 1$.

d. Prove: the IVP $x' = \sum_{m=1}^{\infty} \frac{sin(m \cdot x) \cdot cos(m \cdot t)}{m^{\sqrt{5}}}$, $x(t_0) = x_0$ admits the unique local solution for any $(t_0, x_0) \in \mathbb{R}^2$. Moreover, this solution extends (uniquely) to $x(t) \in C^\omega(\mathbb{R})$.

2. a. Prove: the sums/products in $\mathbb{R}[[\underline{x}]], \mathbb{C}[[\underline{z}]], C^\omega(\mathcal{U}), \mathcal{O}(\mathcal{U})$ are well defined. (Therefore these are commutative rings.) For $C^\omega(\mathcal{U}), \mathcal{O}(\mathcal{U})$ don't forget to check: the product of locally convergent series is locally convergent.

b. Strengthen the statement of Abel's theorem for the power series $\sum a_{\underline{m}} \underline{x}^{\underline{m}}$ to: "If for some $\underline{x}_0 \in \mathbb{R}^n$ the set $\left\{|a_{\underline{m}} \underline{x}_0^{\underline{m}}|\right\}_{\underline{m}}$ is 'sub-exponentially' bounded, i.e. $\lim\limits_{|\underline{m}| \to \infty} \frac{ln(1+|a_{\underline{m}} \underline{x}_0^{\underline{m}}|)}{|\underline{m}|} = 0$, then ...".

c. Suppose the series $\sum a_{\underline{m}} \underline{x}^{\underline{m}}$ converges uniformly on $\mathcal{U} \subset \mathbb{R}^n$. Prove: $\partial_{x_j} \sum a_{\underline{m}} \underline{x}^{\underline{m}} = \sum a_{\underline{m}} \partial_{x_j}(\underline{x}^{\underline{m}})$ and $\int (\sum a_{\underline{m}} \underline{x}^{\underline{m}}) dx_j = \sum a_{\underline{m}} (\int \underline{x}^{\underline{m}} dx_j)$.

d. Define $f(x) = e^{-\frac{1}{x^2}}$ for $x = 0$ and $f(0) = 0$. Prove: $f \in C^\infty(\mathbb{R}^1) \setminus C^\omega(\mathbb{R}^1)$.
Find the radius of convergence of the Taylor series of f at a point $x_o \neq 0$.
(Hint: no long computations are needed.)

e. Suppose $\mathcal{U} \subseteq \mathbb{R}^n$ is path-connected and $f \in C^\omega(\mathcal{U})$ vanishes locally near a point $x_o \in \mathcal{U}$.
Prove: $f = 0$ on the whole $\mathcal{U}$.
Does this hold also for C^∞-functions?

3. The *set of convergence* of a series is defined by $\mathfrak{S} := \{\underline{x} | \ \sum a_{\underline{m}} \underline{x}^{\underline{m}} \text{ converges}\} \subseteq \mathbb{R}^n$.

a. Verify: for $n = 1$ one has $(-R, R) \subseteq \mathfrak{S} \subseteq [-R, R]$, where R is the radius of convergence.

b. Fix some $a, b \in \mathbb{N}$ and find $\mathfrak{S}$ for $\sum_m c_m \cdot (x^a y^b)^m$, here the sequence $c_\bullet$ is bounded and $c_m \not\to 0$.
Among all the open boxes $(-x_0, x_0) \times (-y_0, y_0) \subset \mathfrak{S}$ does there exist "the largest box"?
Is $\mathfrak{S}$ a convex set?

c. (For $n > 1$ we try to establish weaker versions of a.) (Dis)Prove:

i. $\mathfrak{S} \subseteq \overline{Int(\mathfrak{S})}$ (the closure of the interior); (Hint: $f(x_1, x_2) = \frac{x_1}{1-x_2}$)

ii. $\mathfrak{S}$ is of "star-type", i.e. for any $x \in \mathfrak{S}$ the segment $[o, x] \subset \mathbb{R}^n$ lies in $\mathfrak{S}$.

4. Define the distance between two sets $S_1, S_2 \subset \mathbb{R}^n$ by $d(S_1, S_2) := inf\{d(s_1, s_2) | \ s_i \in S_i\}$. Prove:

a. $d(x, S) = 0$ iff $x \in \overline{S}$. (Give an example with $x \notin S$.)

b. If S is closed then $d(x, S) = d(x, s)$ for some $s \in S$. (What can happen if S is not closed?)

c. If $S_1, S_2 \subset \mathbb{R}^n$ are bounded then $d(S_1, S_2) = 0$ iff $\overline{S_1} \cap \overline{S_2} \neq \varnothing$.
Give an example of bounded sets with $S_1 \cap S_2 = \varnothing$ but $d(S_1, S_2) = 0$.
Give an example of closed unbounded sets with $\overline{S_1} \cap \overline{S_2} = \varnothing$ but $d(S_1, S_2) = 0$.

d. If S_1, S_2 are compact then there exist $s_1 \in S_1$, $s_2 \in S_2$ such that $d(s_1, s_2) = d(S_1, S_2)$.