Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 5. Submission date: 01.05.2023

Questions to submit: 1.a. 1.b. 1.d. 2.c. 2.f. 3.a. 4.a. 4.c.
Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. a. (Comparison test) Consider two equations $x' = f_i(t,x)$, $i = 1,2$, and their solutions $x_i(t)$, both defined on $[t_0, t_1)$. Suppose $x_1(t_0) \le x_2(t_0)$. Suppose the bound $f_1(t,x) \le f_2(t,x)$ holds in a neighborhood of the curve $\{(t, x_1(t))\} \subset \mathcal{U}$. Prove: $x_1(t) \le x_2(t)$ on $[t_0, t_1)$.

b. Why is the expansion $Taylor_{t_0}[f]$ presented as $e^{(t_1-t_0)\frac{d}{dt}} f|_{t=t_1}$ and not just $e^{(t-t_0)\frac{d}{dt}} f$?

c. Let $x(t)$ be the (local) solution of $x' = e^{tx^2}$, $x(0) = 0$. Find $Taylor_0[x(t)]$ up to order 6.

d. Prove: the local solution of 1.c extends (uniquely) to $x(t) \in C^\omega(-\infty, \epsilon)$ but explodes (in finite time) on the interval $(\epsilon, 2)$.

2. Let (X, d) be a metric space.

a. Prove: any convergent sequence is a Cauchy sequence, and the limit is unique.

b. Suppose (X, d) is complete. Prove: a subspace $Y \subset X$ is complete iff Y is closed.

c. Give a metric on $X = (-1, 1)$ such that (X, d) is complete. (Does this contradict part b.?)

d. For a closed subset $X \subseteq \mathbb{R}^n$ consider $C^0(X)$, with *sup*-norm.
Prove: $C^0(X)$ is a complete normed space.

e. Suppose d_1, d_2 are equivalent metrics. Prove: (X, d_1) is complete iff (X, d_2) is complete.

f. Give an example of a non-complete metric space and a contractive map without fixed points.

g. Can the assumption in the fixed point theorem be weakened to "$d(\Psi(x),\Psi(y))<d(x,y)$ for $x\neq y$"?

3. a. Define the operator $\Psi : C^0[0,1] \to C^0[0,1]$ by $g \to \int_0^x g(t)dt$.
Prove (for the *max*-norm): $\|\Psi(g)\| < \|g\|$ for $g(0) = 0$ and $g \neq 0$.
Prove: for any $\epsilon > 0$ exists g satisfying: $\|\Psi(g)\| > (1-\epsilon) \cdot \|g\|$. (We did this in the class.)

b. Prove: $x(t) \in C^1$ is a solution of the IVP $x' = f(t,x)$, $x(t_0) = x_0$ iff $x(t)$ is a solution of the integral equation $x = x_0 + \int_{t_0}^t f(s,x)ds$.

c. Write down Picard's approximation $x_3(t)$ for the IVP $x' = 1 + x^2$, $x(0) = x_0$. Prove: $x_k(t) \in C^\omega(\mathbb{R})$ for each $k \in \mathbb{N}$. Does this contradict the explosion of the solution at finite time?
Prove: $x_k(t)$ converges to the solution for $|t| \le \frac{1}{2x_0 + 2\sqrt{1+x_0^2}}$.

4. Let $\{x_k(t)\}$ be Picard's approximations for the IVP $x' = f(t,x)$, $x(t_0) = x_0$, with $|f(t,x) - f(t,y)| \le L \cdot |x-y|$ and $|f(t,x)| \le L \cdot C$ on $\mathcal{U}$. Prove:

a. $|x_{k+1}(t) - x_k(t)| \le \frac{C \cdot L^{k+1} \cdot |t-t_0|^{k+1}}{(k+1)!}$ and $|x(t) - x_k(t)| \le \frac{C \cdot L^k \cdot |t-t_0|^k}{k!}$, for t near t_0.

b. The series $x_m(t) + \sum_{k=m}^{\infty} [x_{k+1}(t) - x_k(t)]$ converges to the solution $x(t)$.

c. If $f \in C^r(\mathcal{U})$, for $r \le \infty, \omega$, then $x_k(t) \in C^{r+1}$ for $k \ge 1$. Moreover, if $x_k(t) \rightrightarrows x(t)$ (uniformly) on $[t_0 - \epsilon, t_0 + \epsilon]$ then $x_k(t)^{(i)} \rightrightarrows x(t)^{(i)}$ on $[t_0 - \epsilon, t_0 + \epsilon]$, for all $i \le r$.

5. Define the function $f(x)$ as $x^2 \cdot sin\frac{1}{x^2}$ for $x < 0$, as $\sqrt{x}$ for $0 < x < 1$ and as $e^{-x^2} \cdot sin(e^{x^3})$ for $x > 1$. (Dis)Prove:

a. f is locally Lipschitz at each point where it is defined.

b. f is Lipschitz on $(-\epsilon, \epsilon) \setminus \{0\}$.

c. f is Lipschitz on $(1-\epsilon, 1+\epsilon) \setminus \{1\}$.

d. f is Lipschitz on $(-\infty, -1)$ and on $(1, \infty)$.