Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 6. Submission date: 7.05.2023

Questions to submit: 1. 2.b. 2.e. 2.f. 3.b. 3.f. 5.a.b.c.
Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. Prove: if $A \in Mat_{n\times n}(\mathbb{R})$ is $\mathbb{C}$-diagonalizable then A is $\mathbb{R}$-conjugate to a (real) block-diagonal matrix, with blocks of size ≤ 2. Moreover, each 2×2 block can be brought to the form: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
(Hint: the non-real eigenvectors of A come in conjugate pairs. But we want a real basis, ...)

2. a. Prove: the functions $A = \sqrt{trace(\bar{A}^t \cdot A)}$, $||A||_{op} := \sup_{||v||\neq 0} \frac{||Av||}{||v||}$ define norms on $Mat_{n\times n}(\mathbb{R})$ and $Mat_{n\times n}(\mathbb{C})$. Moreover, $||A \cdot B||_{op} \leq ||A||_{op} \cdot ||B||_{op}$.

b. Disprove: i. $\|A\|_{op}$ equals the largest eigenvalue of A.
ii. The norm $\| * \|_{op}$ is conjugation-invariant (i.e. $\|A\|_{op} = \|UAU^{-1}\|_{op}$.)

c. Prove: the norms $|| * ||$, $|| * ||_{op}$ are equivalent. Prove: these normed spaces are complete.

d. Review questions 7,8 of homework.0

e. Prove: if $e^{At}e^{Bt} = e^{(A+B)t}$ holds for all $t \in (-\epsilon, \epsilon)$ then $AB = BA$.

f. Prove: if $e^A = \mathbb{1}$ then A is $\mathbb{C}$-diagonalizable. What are the possible eigenvalues?

3. a. Take the unit ball $Ball_1(\mathbb{O})_{op} := \{A|\ ||A||_{op} < 1\} \subset Mat_{n\times n}(\mathbb{C})$.
Prove: if $A \in Ball_1(\mathbb{O})_{op}$ then $\mathbb{1} + A \in GL(n, \mathbb{C})$.

b. For a marix $A \in Ball_1(\mathbb{O})_{op}$ define $ln(\mathbb{1}+A) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}A^k}{k}$. Prove: the series converges absolutely, and the convergence is uniform on compact subsets of $Ball_1(\mathbb{O})_{op}$.

c. Prove: $exp(ln(\mathbb{1}+A)) = \mathbb{1}+A = ln(e^{(\mathbf{1}+A)})$ for every $A \in Ball_1(\mathbb{O})$. (No long computations are needed here.)

d. Take the subset $Mat_{2\times 2}(\mathbb{C}) \supset \Sigma := \{A|\ e^A = \mathbb{1},\ det(t\mathbb{1} - A) = t^2 + 4\pi^2\}$. Identifying $Mat_{2\times 2}(\mathbb{C}) \cong \mathbb{C}^4$ prove: Σ is defined by one linear and one quadratic equation. (Hint: observe that any matrix in Σ must be diagonalizable.)
Conclude: Σ contains a two-parametric family of matrices. (Therefore, while the map exp is locally invertible near 0, it is highly non-injective globally.)

e. Prove: if $AB = BA$ and $A, B, A + B + AB \in Ball_1(\mathbb{O})_{op}$ then $ln\big[(\mathbb{1} + A)(\mathbb{1} + B)\big] = ln(\mathbb{1} + A) + ln(\mathbb{1} + B)$. In particular, $ln[(\mathbb{1} + A)^k] = k \cdot ln(\mathbb{1} + A)$ for every $k \in \mathbb{Z}$.

f. Compute $\frac{d}{dt}ln(\mathbb{1} + At)$. (Do this in two ways, as we did for $\frac{d}{dt}e^{At}$ in the lecture.)

4. Define the functions $Mat_{n\times n}(\mathbb{C}) \overset{sin,cos}{\to} Mat_{n\times n}(\mathbb{C})$ via the Taylor expansion of sin, cos. Prove:

a. These series converge absolutely, the convergence is uniform on bounded subsets of $Mat_{n\times n}(\mathbb{C})$.

b. Prove: $e^{iA} = cos(A) + i \cdot sin(A)$. $\quad cos(A) = \frac{e^{iA}+e^{-iA}}{2}$. $\quad sin(A) = \frac{e^{iA}-e^{-iA}}{2i}$.

c. Prove: $sin^2(A)+cos^2(A) = \mathbb{1}$. $\quad$ If $AB = BA$ then $sin(A+B) = \cdots$, $\quad cos(A+B) = \cdots$.

d. Compute $\frac{d}{dt}cos(At)$ and $\frac{d}{dt}sin(At)$. (Do this in two ways, as we did for $\frac{d}{dt}e^{At}$ in the lecture.)

5. In the following cases (without solving the equations):

i. Identify the equilibria points. When are these points (un)stable nodes/saddles?

ii. For which λ are there (un)bounded/periodic solutions?

iii. For the cases a. and b. draw the phase portraits.

Now write down the general (real) solutions, and verify the previously obtained properties.

a. $x' = y$, $y' = \lambda \cdot x$. (Distinguish between the cases $\lambda > 0$, $\lambda < 0$.)

b. $x' = \lambda x + y$, $y' = \lambda y$.

c. $\underline{x}' = A \cdot \underline{x}$ for $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.