Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 7. Submission date: 14.05.2023

Questions to submit: 1.b.d. 2.a.b.c.e. 3.c.

Homeworks must be either typed (e.g. in Latex) or written in readable handwriting and scanned in readable resolution.

1. Consider the system of differential equations $\underline{x}' = A \cdot \underline{x}$, $A \in Mat_{n\times n}(\mathbb{R})$. Prove:

a. If $\underline{x}(t)$ is a solution then all its derivatives are solutions.

b. If $A = A^t$ then there are no (non-constant) periodic solutions.

c. If $A = -A^t$ then the space of solutions is spanned by periodic solutions. Does this imply that every solution is periodic?

d. If A is of odd size then there exists an unbounded solution.

e. What is the necessary and sufficient condition on A to ensure $\lim\limits_{t\to\infty} \underline{x}(t) = \underline{0}$ for each solution?

f. If A is $\mathbb{R}$-diagonalizable and the eigenvalues have the same sign then $\underline{x} = 0$ is a nodal point. (Attracting or repelling)

g. The solutions are analytic in the initial data, $x(t, t_0, x_0, A) \in C^\omega(\mathbb{R}_t \times \mathbb{R}_{t_0} \times \mathbb{R}^n_{\underline{x}_0} \times Mat_{n\times n}(\mathbb{R}))$.

h. The set of equilibrium points is a vector subspace of $\mathbb{R}^n$. (What is the dimension?)

2. a. Write down the general (real) solutions of the equations

i. $x^{(n+2)} \pm b^2 x^{(n)} = 0$. ii. $x^{(4)} + 2x^{(3)} + 2x^{(2)} = 0$.

b. Find the condition on a, b to ensure: 0 is a stable equilibrium point of $x^{(2)} + a\cdot x' + b\cdot x = 0$.

c. Find an equation $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = 0$ of minimal possible order whose general solution contains the functions $sin(2t)\cdot e^t$, $cos(2t)\cdot e^{2t}$, t^2. Explain why is this order minimal. Prove: the coefficients $\{a_j\}$ are uniquely determined.

d. Fix a finite set of pairwise distinct complex numbers $\{\lambda_k\}$ and some natural numbers $\{p_k\}$. Prove: the functions $\{t^j \cdot e^{\lambda_k t}\}_{\substack{1\le k\le n\\ 0\le j\le p_k}}$ are $\mathbb{C}$-linearly independent.

(Can you do this in several different ways?)

e. Prove the $\mathbb{R}$-version of part d., about the $\mathbb{R}$-linear independence of the functions $\{t^j e^{a_k t}\cdot cos(b_k\cdot t)\}$, $\{t^j e^{a_k t}\cdot sin(c_k\cdot t)\}$.

3. Consider the ODE $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = 0$ and the equation $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$.

a. (Dis)prove: if all the λ-roots are imaginary, simple then every solution of this ODE is periodic.

b. (Dis)prove: if all the λ-roots are real-positive then every solution of this ODE is monotonic.

c. Prove: for any tuple $(a_{n-1}, \ldots, a_1)$ there exists a finite subset $S \subset \mathbb{R}$ such that for $a_0 \in \mathbb{R}\setminus S$ the space of solutions is spanned by exponents. (Hint: when does a polynomial have a multiple root?)

This phenomenon causes statements like "The solutions $t^j e^{\lambda t}$ never appear in laboratory".

4. Prove: any (finite) system of ODE's, $\underline{f}(t, \underline{x}, \underline{x}', \ldots, \underline{x}^{(k+1)}) = \underline{0}$, is equivalent to a system of 1'st order ODE's, $\underline{F}(t, \underline{y}, \underline{y}') = \underline{0}$. (Namely, every solution of $\underline{f}$ leads to a solution of $\underline{F}$ and vice versa.)

Moreover, if the initial system $\underline{f}(\ldots)$ is in the normal form/autonomic/linear/polynomial then the resulting system $\underline{F}(\ldots)$ is of this type as well.