Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 8. Submission date: 27.05.2023

Questions to submit: 1 2.a. 2.c. 3.a. 3.c. 4.b. 5.b.

Either typed or in readable handwriting and scanned in readable resolution.

1. Write the general solution of the equation $\underline{x}' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix} \cdot \underline{x} + \begin{bmatrix} 3e^t \\ 0 \\ 3e^{-t} \end{bmatrix}$.

2. a. Prove: if the function $g(\underline{x}) > 0$ is continuous then the systems $\underline{x}' = \underline{f}(\underline{x})$ and $\underline{x}' = g(\underline{x}) \cdot \underline{f}(\underline{x})$ have the same phase potraits. (What happens for $g < 0$?)

b. Prove: the phase curves of the system $\underline{x}' = \underline{f}(\underline{x})$, $\underline{f} \in C^1(\mathcal{U})$ for $\mathcal{U} \subseteq \mathbb{R}^n$, cover the whole $\mathcal{U}$ and either coincide or do not intersect.

c. Consider the system $x' = sin(x) \cdot (e^{y^2} + x^4)$, $y' = sin(cos(y)) \cdot (e^{x^2} + y^3)$.

i. Find the equilibria points.

ii. Prove: there exist infinity of phase curves that are parallel to $\hat{y}$-axis. Moreover, each of these curves is an open interval of length$< \pi$. (And the same for $\hat{x}$-axis.)

iii. Prove: any local solution extends (uniquely) to the global solution $x(t), y(t) \in C^\omega(\mathbb{R})$.

3. Consider the system $\underline{x}' = \underline{f}(t, \underline{x})$, with $\underline{f} \in C^r((a, b) \times \mathbb{R}^n)$. We have proved: If $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g(t) \cdot (1 + ||\underline{x}||^2)$ then any solution extends to $C^{r+1}(a, b)$.

a. Instead of $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g(t) \cdot (1 + ||\underline{x}||^2)$ one could take the condition $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g_0(t) + g_1(t) \cdot ||\underline{x}|| + g_2(t) \cdot ||\underline{x}||^2$, for some g_0, g_1, g_2. Prove: this condition is not essentially weaker. Namely, this condition holds for some g_0, g_1, g_2 iff the previous condition holds for some g.

b. Suppose the bound $|\underline{x} \cdot \underline{f}(t, \underline{x})| \le g(t) \cdot (1 + \phi(||\underline{x}||^2))$ holds for some function $g(t)$ and a function $\phi(y) \ge 0$ satisfying: $\int_0^\infty \frac{dy}{1+\phi(y)} = \infty$. Prove: any solution extends to $C^{r+1}(a, b)$. For which function ϕ do we get the criterion proved in the class? For which functions ϕ we get a stronger criterion?

c. Consider the equation $x^{(n)} = f(t, x, \dots, x^{(n-1)})$, where $f \in C^r((a, b) \times \mathbb{R}^n)$. Denote $\underline{y} = (y_0, \dots, y_{n-1})$. Suppose the bound $|y_{n-1} \cdot f(t, \underline{y})| \le g(t) \cdot (1 + |\underline{y}|^2)$ holds in $(a, b) \times \mathbb{R}^n$. Prove: any local solution extends to a global one, $x(t) \in C^{r+1}(a, b)$.

4. a. Verify: $e^{\underline{a} \cdot \nabla} f(\underline{x}) = f(\underline{x} + \underline{a})$, here $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$.

b. Write down the Taylor expansion of a solution $\underline{x}(t)$ of $\underline{x}' = A \cdot \underline{x}$ using the general formula for Taylor power series, as was given in the class. Verify that you get $\underline{x}(t) = e^{A(t-t_0)} \cdot \underline{x}_0$.

c. Let $\underline{x}(t)$ be the solution of $\underline{x}' = A(t) \cdot \underline{x}$, $\underline{x}(t_0) = \underline{x}_0$. Compute the Taylor expanion of $\underline{x}(t)$ up to order 3. (Attention, the matrices $A(t)$, $A'(t)$ do not necessarily commute.)

5. a. Let $A(t) \in Mat_{n \times n}(C^r(\mathbb{R}))$, for $1 \le r \le \infty, \omega$. Prove: any local solution of $\underline{x}' = A(t) \cdot \underline{x}$ extends (uniquely) to a global solution $\underline{x}(t) \in C^{r+1}(\mathbb{R})$.

b. Let $\underline{x}(t)$, $\underline{y}(t)$ be solutions of $\underline{x}' = A(t) \cdot \underline{x}$. Prove: $||\underline{x}(t) - \underline{y}(t)|| \le ||\underline{x}(t_0) - \underline{y}(t_0)|| \cdot e^{\int_{t_0}^t ||A(s)||_{op} ds}$.

c. Consider the system $\underline{x}' = f(t, \underline{x})$ for $f \in C^0(\mathcal{U})$. Suppose $|(\underline{x} - \underline{y}) \cdot (\underline{f}(t, \underline{x}) - \underline{f}(t, \underline{y}))| \le g(t) \cdot e^{||\underline{x} - \underline{y}||^2}$ in $\mathcal{U}$. Prove: any solutions $\underline{x}(t)$, $\underline{y}(t) \in C^1(a, b)$ satisfy $|\underline{x}(t) - \underline{y}(t)|^2 \le |\underline{x}(0) - \underline{y}(0)|^2 - ln[1 - e^{|\underline{x}(0) - \underline{y}(0)|^2} \cdot \int_{t_0}^t g(s) ds]$. (We assume here $e^{|\underline{x}(0) - \underline{y}(0)|^2} \cdot \int_{t_0}^t g(s) ds < 1$.)