Ordinary differential equations for Math

(201.1.0061. Spring 2023. Dmitry Kerner)

Homework 9. Submission date: 03.06.2023

Questions to submit: 1.a. 2.i. 2.ii. 2.iv. 2.v. 3.b. 4.c. 5.b. 5.c.

Either typed or in readable handwriting and scanned in readable resolution.

1. Consider the equation $\underline{x}' = A\cdot\underline{x} + e^{\mu t}t^k\cdot\underline{b}$, here $\underline{b}$ is a constant vector. Prove:

- **a.** If μ is not an eigenvalue of A then there exists a solution of the form $e^{\mu t}\cdot\underline{g}(t)$, where the entries of $\underline{g}$ are polynomials of degree$\leq k$.
- **b.** If μ is an eigenvalue of A then there exists a solution of the form $e^{\mu t}\underline{g}(t)$, where the entries of $\underline{g}$ are polynomials of degree$\leq k + Jord.Size.(A)$, here $Jord.Size.(A)$ is the size of the maximal Jordan cell of A.

2. Prove: **i.** $det(e^A)=e^{trace(A)}$. **ii.** $\det[\mathbb{1}+\epsilon A]=1+\epsilon\cdot trace(A)+O(\epsilon^2)$. **iii.** $\|e^A\|_{op}\leq e^{\|A\|_{op}}$.
iv. $e^A=\lim\limits_{k\to\infty}(\mathbb{1}+\frac{A}{k})^k$. **v.** If $A(t)\in GL(n,C^1(a,b))$ then $(A(t)^{-1})'=-A(t)^{-1}A'(t)A(t)^{-1}$.

3.
- **a.** Write the general solution of the system $x' = \frac{x}{1+t^2} + y\cdot sin(2t)$, $y' = y\cdot cos(t)$.
- **b.** Write the general solution of the equation $(\frac{d}{dt}-a_1(t))\circ(\frac{d}{dt}-a_2(t))x=0$, $a_1(t), a_2(t)\in C^1(a,b)$.
- **c.** Find the full Taylor series of the solution of $x'' + t^p\cdot x = 0$, $\quad p\in\mathbb{N}$, $\quad x(0) = 0$, $x'(0) = 1$.

4.
- **a.** Suppose the matrices $A(t)$ and $\int_{t_0}^t A(s)ds$ commute for each $t\in(a,b)$.
  Prove: the (unique) solution of $\underline{x}' = A(t)\cdot\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$ is given by $\underline{x}(t) = e^{\int_{t_0}^t A(s)ds}\underline{x}_0$.
- **b.** Let $\{A_j\}$ be some constant pairwise commuting matrices. Let $\{g_j(t)\}$ be $C^0(a,b)$.
  Solve the system $\underline{x}' = (\sum g_j(t)A_j)\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$.
- **c.** Prove that the assumption in **a.** implies: the matrices $A(t)'$ and $\int_{t_0}^t A(s)ds$ commute for each $t\in(a,b)$.
- **d.** Give an example of equation $\underline{x}' = A(t)\cdot\underline{x}$ for which $e^{\int_{t_0}^t A(s)ds}\underline{x}_0$ is not a solution.

5. Equations of type $t^n x^{(n)} + a_{n-1}t^{n-1}x^{(n-1)} + \cdots + a_1 t x' + a_0 x = 0$ are called Euler-Cauchy equations. They are used e.g. in physics and in finance.

- **a.** One approach to solve is by time rescaling. Prove: the substitution $t = e^\tau$ transforms the Euler-Cauchy equation into a linear ODE with constant coefficients.
- **b.** Write the general solution for the ODE $t^2x'' + tx' + a_0x = 0$.
- **c.** In the general case prove: the characteristic polynomial of the obtained ODE with constant coefficient is $L(\lambda) := \lambda(\lambda-1)\cdots(\lambda-n+1) + a_{n-1}\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots + a_0$.
- **d.** Prove: if one presents $L(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_0$ then the initial equation can be presented as $(t\frac{d}{dt})^n x + b_{b-1}(t\frac{d}{dt})^{n-1}x + \cdots + b_0 x = 0$.
- **e.** Conclude: for $t\neq 0$ the space of solutions of a Euler-Cauchy equation is spanned by the functions of type $\{ln(t)^{k_j}\cdot t^{\lambda_j}\}$, with $k_j\in\mathbb{N}$. Here the function t^λ for $\lambda\in\mathbb{C}$ is defined by $t^\lambda := e^{\lambda\cdot ln(t)} = |t|^{Re(\lambda)}\big(cos(Im(\lambda)ln(t)) + i\cdot sin(Im(\lambda)ln(t))\big)$.