Introduction to Differential Topology, 201.2.7061
Moed A, 05.02.2024, three hours.

(Lecturer: Dmitry Kerner)

No auxiliary material is allowed. Do not write in red color.

Solve all the questions. (15 points each, 105 points in total)

1. Take a C^2-function germ $(\mathbb{R}^n, o) \xrightarrow{f} (\mathbb{R}, o)$. Prove: the condition "$[f'|_o = 0$ and $f''|_o$ is non-degenerate]" is preserved under (C^2) coordinate-changes.

In this case prove: the signature n_+, n_- of $f''|_o$ is preserved under (C^2) coordinate-changes.

2. A subset $X \subset \mathbb{R}^{n+1}$ is defined by the equation $f(x) = 0$, for a scalar valued C^1-function. Suppose f' has no zeros on X. Prove: X is orientable.

3. Fix two manifold-germs, $(X_1, o), (X_2, o) \subset (\mathbb{R}^N, o)$, suppose their intersection is smooth. (Dis)prove: if $T_{(X_1 \cap X_2, o)} = T_{(X_1,o)} \cap T_{(X_2,o)}$, then $(X_1, o) \pitchfork (X_2, o)$.

4. Prove: the torus $S^1 \times S^1$ is not embeddable into $\mathbb{R}^2$.

5. Fix an embedding $S^1 \hookrightarrow S^1 \times S^1$. Prove: the normal bundle $\mathcal{N}(S^1/S^1 \times S^1)$ is trivial.

6. A manifold X is compact, orientable, with no boundary. Suppose Y is connected. Suppose a map $X \xrightarrow{f} Y$ has $deg(f) \neq 0$. Prove: f is surjective.

7. Suppose a compact manifold X (with $\partial X = \varnothing$) is contractible. Prove: X=one point.

Good Luck!