

# Introduction to Differential Topology, 201.2.7061

Moed B, 27.02.2024, three hours.

(Lecturer: Dmitry Kerner)

No auxiliary material is allowed.

Solve all the questions. (105 points in total)

Do not write in red color!

1. [15points] Prove: every immersion  $\mathbb{R}^1 \rightarrow \mathbb{R}^1$  is a diffeomorphism onto its image.
2. [15points] Prove: every connected manifold is path-connected.
3. [15points] Let  $X \xrightarrow{f} \mathbb{R}$  be a Morse function on a compact manifold (without boundary). Prove: for any deformation  $\{f_t\}$  there exists  $0 < t_o \ll 1$  such that  $X \xrightarrow{f_t} \mathbb{R}$  is Morse function for  $|t| \leq t_o$ .
4. [15points] Let  $X \xrightarrow{f} \mathbb{R}$ , where  $X$  is a manifold without boundary, and 0 is a regular value. Prove:  $f^{-1}(\mathbb{R}_{\geq 0}) \subset X$  is a manifold with boundary and  $\partial(f^{-1}\mathbb{R}_{\geq 0}) = f^{-1}(0)$ .
5. [15points] Take a manifold  $X = \mathcal{U}_1 \cup \mathcal{U}_2$ , with  $\mathcal{U}_1, \mathcal{U}_2$  open and orientable. Suppose  $\mathcal{U}_1 \cap \mathcal{U}_2$  is path-connected. Prove:  $X$  is orientable.
6. [15points] Take an embedding  $S^1 \xrightarrow{\phi_o} X$  into a smooth surface. Prove: if  $X$  is orientable, then there exists deformation of  $\phi_o$  satisfying:  $\phi_o(S^1) \cap \phi_\epsilon(S^1) = \emptyset$  for  $\epsilon \neq 0$ .
7. [15points] Compute the degree of the antipodal map  $S^n \xrightarrow{-Id} S^n, x \rightarrow -x$ .

*Good Luck!*