Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

Homework 0

Not for submission. It is important to solve the questions before the first lecture.

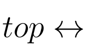
Notations and conventions.

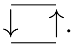
- Subsets of $\mathbb{R}^N$ are considered with the induced topology. Namely, the open subsets of $X \subset \mathbb{R}^N$ are all of the form: $\mathcal{U} \cap X$, for open subsets $\mathcal{U} \subset \mathbb{R}^N$.
- $Ball_\epsilon$ is an open ball of radius $\epsilon > 0$ in $\mathbb{R}^N$.
- We use multivariables, e.g. $\underline{x} = (x_1, \dots, x_N)$, $\underline{f} = (f_1, \dots, f_l)$. Sometimes we write just x, f.
- The matrix of partial derivatives of a function $\mathbb{R}^N \xrightarrow{f} \mathbb{R}^l$ is $f' := \frac{\partial f}{\partial x} := [\frac{\partial(f_1 \dots f_l)}{\partial(x_1 \dots x_N)}] \in Mat_{l \times N}$

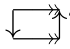
1. a. Take the map $\mathbb{R} \xrightarrow{\phi} \mathbb{R}^3$, $\phi(t) = (t, t^2, t^3)$. Denote the image-curve by $X \subset \mathbb{R}^3$. Write the equations of the projections of X onto the planes (x, y), (y, z), (x, z).
   b. Draw the subsets in $\mathbb{R}^3$ defined by equations: $\{xyz = 0\}$, $\{xz = 0 = yz\}$. In each case, which points you would call "smooth"? What is your guess for the dimension of these sets?
   c. Write the defining equation of the surface obtained by the full (2π) rotation of the curve $\{y^2 + z^2 = 1,\ x = 0\} \subset \mathbb{R}^3$ around the $\hat{z}$ -axis.
   d. Write the equation of the surface obtained by the full (2π) rotation of the curve $\{z^2 - y^2 = 1,\ x = 0\} \subset \mathbb{R}^3$ around the $\hat{z}$ -axis. Draw/identify the surface.
   e. Write the equation of the surface obtained by the full (2π) rotation of the curve $\{(y - R)^2 + z^2 = r^2,\ x = 0\} \subset \mathbb{R}^3$ around the z -axis. Here $0 < r < R$. Draw/identify the surface. (A torus)
   f. Define the surface $X_\epsilon \subset \mathbb{R}^3$ by the equation $z^2 = x^2 + y^2 + \epsilon$, for each $\epsilon \in \mathbb{R}$. Draw X_ϵ for $\epsilon > 0$, $\epsilon = 0$, $\epsilon < 0$. How X_ϵ changes (topologically) when $\epsilon \to 0^+$ and $\epsilon \to 0^-$?
   g. Intersect the torus of q.1.e. by the planes $\{x = c\}$, for $c \in [-R - r, R + r]$. For which values of c you get (topologically) S^1? When do you get $S^1 \coprod S^1$? Describe the other ("degenerate") sections.

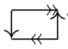
2. a. A subset $X \subset \mathbb{R}^N$ is defined by a system of equations $\{f_i(x) = 0\}_i$ (finite or infinite). Here $f_i \in C^0(\mathbb{R}^N)$. Prove: X is a closed subset.
   b. Consider open subsets of the sphere $S^2 \subset \mathbb{R}^3$. Project these to the plane $\mathbb{R}^2_{xy}$. Are the images always open?
   c. (Dis)Prove: finite products, unions, intersections of compact sets in $\mathbb{R}^N$ are compact.
   d. (Dis)Prove: $X \subset \mathbb{R}^n$ is compact iff all its projections onto coordinate hyperplanes ($\{x_j = 0\} \subset \mathbb{R}^n$) are compact.
   e. Suppose a function $\mathbb{R}^N \supseteq X \xrightarrow{f} Y \subset \mathbb{R}^m$ is continuous, bijective, and X is compact. Prove: f^{-1} is also continuous. What can happen when X is non-compact?
   f. Let $X \subset \mathbb{R}^n$ and suppose $a \in int(X)$, $b \in int(\mathbb{R}^n \setminus \overline{X})$. Prove: any path from a to b intersects ∂X.

3. a. Consider the map $\mathbb{C} \ni z \to exp(z) \in \mathbb{C}$. Identify $\mathbb{C} \cong \mathbb{R}^2$, by $z \to (Re(z), Im(z))$, and take the corresponding real function, $\mathbb{R}^2 \xrightarrow{[exp]_\mathbb{R}} \mathbb{R}^2$. Prove: this function is C^∞-invertible locally at each point of $\mathbb{R}^2$. Is this function globally invertible?
   b. Give an example of a function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ such that: $f'|_0 = 0$, but f is C^0-globally invertible.
   c. Prove: if $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is C^1 and $f'|_0 = 0$ then f is not C^1-locally invertible at $x = 0$.
   d. Take a C^1-function $Ball_\epsilon \xrightarrow{f} \mathbb{R}^l$. Suppose $rank[f'|_o] = k$ at a point $o \in Ball_\epsilon$. Prove: $rank[f'|_x] \geq k$ for x close to o. (Namely: there exists an open neighborhood $o \in \mathcal{U} \subseteq Ball_\epsilon$ such that ...)

4. a. Take a page and glue its sides: , i.e. $left \leftrightarrow right$ and $top \leftrightarrow bottom$. Identify this surface.

b. Glue the sides $l \leftrightarrow (-r)$, with reversed orientation, . Identify the surface. Wiki: Möbius strip.

c. Which surface you get by gluing the sides $l \leftrightarrow (-r)$ and $t \leftrightarrow b$, i.e. ?
(Can you do this in $\mathbb{R}^3$?) Wiki: "Klein bottle".

d. Which surface you get by gluing the sides $l \leftrightarrow (-r)$ and $t \leftrightarrow (-b)$, i.e. ?
Wiki: "Real projective plane".

5. a. Find the equation of the tangent plane to the standard sphere $S^{n-1} \subset \mathbb{R}^n$, at a point $p \in S^{n-1}$.
b. Let $S^1 \times S^1 = \{\underline{x} |\ x_1^2 + x_2^2 = 1 = x_3^2 + x_4^2\} \subset \mathbb{R}^4$. Find the equations for the tangent plane to this subset at a point $\underline{x}_0$. (e.g. for $x_{2,0}, x_{4,0} > 0$ present $S^1 \times S^1$ as the graph of a function.)
c. Consider the curve $C = \{f_1(\underline{x}) = 0 = f_2(\underline{x})\} \subset \mathbb{R}^3$, where f_1, f_2 are C^1 and $rank(f'|_{\underline{x}_0}) = 2$. Prove: the tangent line to C at $\underline{x}_0$ is given by $\{\underline{x} |\ \underline{x} - \underline{x}_0 \in ker(f'|_{\underline{x}_0})\}$.
d. Prove: $S^{n-1} := \{\underline{x} |\ |\underline{x}| = r\} \subset \mathbb{R}^{n>1}$ is path connected. Do this in as many ways as you can, e.g.: by considering the polar coordinates in $\mathbb{R}^n$; by presenting S^{n-1} as the union of two graphs of continuous functions; by intersecting S^{n-1} with a hyperplane.

6. Write down the definition of "a local C^∞-coordinate change of $\mathbb{R}^n$ at the origin $o \in \mathbb{R}^n$," and "a global C^∞-coordinate change of $\mathbb{R}^n$."
a. Rectify the surface $X = \{z = x^2 + y^2\} \subset \mathbb{R}^3$ globally. Namely, find a global C^∞-coordinate change on $\mathbb{R}^3$ that sends X to a plane.
b. Take the unit sphere, $S^n = \{x |\ \|x\| = 1\} \subset \mathbb{R}^{n+1}$. Fix any point $p \in S^n$. Rectify S^n locally at p. Namely, find a local coordinate change that preserves p and (locally) sends S^n to the tangent plane of S^n at p.
c. Let $X := \{(x, y) |\ f(x, y) = 0\} \subset \mathbb{R}^2$, for $f \in C^\infty(\mathbb{R}^2)$. Suppose $f(o) = 0$ and $f'|_o \neq 0$. Prove: X can be locally rectified (at o) to its tangent line.
d. Take the curve $X = \{x \cdot y = 0\} \subset \mathbb{R}^2$. Prove: X is not locally homeomorphic at the origin o to $\mathbb{R}^1$. Namely, no open neighborhood $o \in \mathcal{U} \subset X$ is homeomorphic to $(0, 1) \subset \mathbb{R}^1$.
Prove: if $f(x, y) = 0$ is *any* defining equation of X (not necessarily $f(x, y) = x \cdot y$), then $grad(f)|_o = \vec{0}$.
e. Prove: the curve $X = \{y^2 = x^3\} \subset \mathbb{R}^2$ is homeomorphic to $\mathbb{R}^1$.
Prove: if $f(x, y) = 0$ is *any* defining equation of X, then $grad(f)|_o = \vec{0}$.
Conclude: there exists no local diffeomorphism of $\mathbb{R}^2$ that sends X to a line.

7. a. Suppose $\mathbb{R}^n \supseteq \mathcal{U}_f \xrightarrow{f} \mathbb{R}^n$ is invertible and both f, f^{-1} are differentiable. Prove: $f'|_{\underline{x}_0} \cdot (f^{-1})'|_{f(\underline{x}_0)} = 1\!\!1$.
b. Prove: if $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^1$ is differentiable and $f(o) = 0$ then $f(\underline{x}) = \sum\limits_{i=1}^{n} x_i g_i(\underline{x})$, for some continuous functions $\{g_i\}$. (Hint: define $h_{\underline{x}}(t) = f(t \cdot \underline{x})$ and note $f(x) = \int\limits_0^1 h'_{\underline{x}}(t)dt$.)
c. Suppose $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}^n$ is C^1. Suppose the equation $f(\underline{x}) = \underline{y}_0$ has k solutions, $\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}$ and they satisfy $\{det[f'|_{\underline{x}^{(i)}}] \neq 0\}_i$. Prove: for some $\epsilon > 0$ and any $\underline{y} \in Ball_\epsilon(\underline{y}_0)$ the equation $f(\underline{x}) = \underline{y}$ has at least k solutions.

8. Establish the normal form of a C^k-function, $k \geq 1$, $\mathbb{R}^n \supseteq \mathcal{U}_f \xrightarrow{f} \mathbb{R}^m$:
a. If $m \leq n$ and $rank[f'|_p] = m$, then in some local (C^k) coordinates at $p \in \mathbb{R}^n$ the function is: $f(\underline{x}) = f(p) + (x_1, \ldots, x_m)$. (It is better to start with the case $m = 1$.)
b. If $m > n$ and $rank[f'|_p] = n$, then in some local (C^k) coordinates at $p \in \mathbb{R}^n$ and at $f(p) \in \mathbb{R}^m$ the function is: $f(\underline{x}) = (x_1, \ldots, x_n, 0, \ldots, 0)$.