Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

Homework 1. Submission date: 18.11.2024

Questions to submit: 2.a, 2.b, 2.c, 3.a, 4.c.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. We have defined the germs of sets, (X, x_o), and of maps, $(X, x_o) \xrightarrow{f} (Y, y_o)$, via equivalence relations. Verify: these are indeed equivalence relations.
   b. Prove: any (non-empty) open subgerm of $(\mathbb{R}^n, o)$ coincides with the germ $(Ball_\epsilon(o), o)$.
   c. Take the germ of a function $(\mathbb{R}^n, o) \xrightarrow{f} \mathbb{R}^1$. Write the full definition for the condition "$f \geq 0$".
   d. Define the basic operations on finite number of germs, $\cap_i(X_i, o) := (\cap_i X_i, o)$, $\cup_i(X, o) := (\cup_i X_i, o)$, $(X, o) \setminus (Y, o) := (X \setminus Y, o)$, $\prod_i(X_i, 0) := (\prod_i X_i, o)$. Verify: the results are well defined, i.e. do not depend on the choice of representatives.
   What happens in the infinite case?
   e. Similarly, define the basic operations on (a finite number of) function germs: $\sum_i f_i$ and $\prod_i f_i$. Verify: the results are well defined.
   f. Let $0 \leq r \leq \infty$. Denote by $C^r(\mathbb{R}^N, o)$ the set of germs of C^r-functions. Verify: the derivatives at the origin, $f^{(j)}|_o$, $j = 0, \dots, r$, are well defined, i.e. depend only on the germ of f. For $r = \infty$ verify: the full Taylor expansion, $Taylor_o[f]$, depends only the germ of f.

2. Take a C^r-germ $(r \geq 1)$: $(\mathbb{R}^n, o) \xrightarrow{f} (\mathbb{R}^m, o)$. Prove (using the implicit function theorem):
   a. (The normal form of a submersion, $m \leq n$) If $rank[f'|_o] = m$, then in some (C^r) coordinates on $(\mathbb{R}^n, o)$ (and the standard coordinates on $(\mathbb{R}^m, o)$) the function is: $f(\underline{x}) = (x_1, \dots, x_m)$.
   b. (The normal form of an immersion, $m \geq n$) If $rank[f'|_o] = n$, then in some local (C^r) coordinates on $(\mathbb{R}^m, o)$ (and the standard coordinates on $(\mathbb{R}^n, o)$) the function is: $f(\underline{x}) = (x_1, \dots, x_n, 0, \dots, 0)$.
   c. Deduce the open mapping theorem: if $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}^m$ is C^1, $m \leq n$, and $rank[f'] = m$ everywhere on $\mathcal{U}$ then f sends open sets to open sets.
   d. Suppose $(\mathbb{R}^n, o)$ and $(\mathbb{R}^m, o)$ are C^1-diffeomorphic. Prove: $n = m$.
   e. Suppose $(\mathbb{R}^n, o) \xrightarrow{f} (\mathbb{R}^N, o)$ is a diffeomorphism onto its image. Prove: $rank[f'|_o] = n$.
   f. Let $(\mathbb{R}^n, o) \xrightarrow{f} (\mathbb{R}^N, o)$ be C^1 with $rank[f'|_o] = k$. Prove: $rank[f'|_x] \geq k$ for x close to o.
   (Namely: there exists an open neighborhood $o \in \mathcal{U} \subset \mathbb{R}^n$, such that ...)

3. Prove: the following sets are C^∞-submanifolds. Compute their dimensions.
   a. $X_n = \{(x, y) |\ \|x\| = \|y\| = 1\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Prove: $X_n \xrightarrow[\sim]{C^\infty} S^{n-1} \times S^{n-1}$.
   b. $\{(x, y, z) |\ x^2 + y^2 = 0,\ x^2 + y^2 + z^2 = 2x\} \subset \mathbb{R}^3$.
   c. $\{(x_1, \dots, x_4) |\ \|x\| = 1,\ x_1 x_2 + x_3 x_4 = 0\} \subset \mathbb{R}^4$.

4. a. Prove: the germ $(C, o) = \{(x, y) |\ x \cdot y = 0\} \subset (\mathbb{R}^2, o)$ is not the germ of a C^0-manifold.
   (i.e. (C, o) is not homeomorphic to $(\mathbb{R}^n, o)$ for any n.)
   b. Prove: $(C, o) = \{(x, y) |\ y = |x|\} \subset (\mathbb{R}^2, o)$ is not the germ of a C^1-submanifold.
   Prove: $\mathbb{R}^1 \ni t \to (t^7, |t|^7) \in \mathbb{R}^2$ is a C^6-parametrization of C. Any contradiction?
   c. Define the curve $C \subset \mathbb{R}^2$ by the parametrization $\mathbb{R}^1 \ni t \to (t^3, t^5) \in \mathbb{R}^2$. Prove: C is a C^1-submanifold. Give an (explicit) non-degenerate C^1-parametrization of C.
   Prove: C is not a C^2-submanifold.
   d. For each $r \geq 1$ give an example of C^r-submanifold in $(\mathbb{R}^2, o)$ that is not a C^{r+1}-submanifold.
   e. Prove: the dimension of a path-connected manifold is well defined.
   (i.e. $dim_p X$ does not depend on p)