

Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

Homework 2. Submission date: 24.11.2024

Questions to submit: 1.b. 1.f. 2.a. 2.b. 2.e. 4.a. 4.c.

(Either typed or in readable handwriting and scanned in readable resolution.)



1.
 - a. Give a C^∞ -diffeomorphism $\mathbb{R}^n \supset \text{Ball} \xrightarrow{\sim} \mathbb{R}^n$.
 - b. Give a C^∞ -diffeomorphism $\mathbb{R}^n \setminus \{o\} \xrightarrow{\sim} \{x \mid 1 < \|x\| < 2\}$.
 - c. The trefoil knot in \mathbb{R}^3 can be defined by the parametrization: $x(t) = \sin(t) + 2\sin(2t)$, $y(t) = \cos(t) - 2\cos(2t)$, $z = -\sin(3t)$. Prove: the trefoil is C^∞ -diffeomorphic to $S^1 \subset \mathbb{R}^2$.
 - d. A theorem in the class gave five equivalent definitions of smoothness. We have proved only: $3 \Rightarrow 1, 2, 4, 5$. Prove the other implications.
 - e. Take $X = \{(x, y) \mid f(x, y) = 0\} \subset \mathbb{R}^2$ for $f(x, y) = y^2$. Thus X is a submanifold, but f' vanishes at each point of X . Does this contradict that theorem?
 - f. Take a C^r -map of manifolds, $X \xrightarrow{f} Y$. Prove: its graph, $\Gamma_f := \{(x, f(x))\} \subset X \times Y$, is a C^r -submanifold.
 - g. Consider the subset of \mathbb{R}^3 defined by the equations $x^3 + y^3 + z^3 = 1$, $2x^6 + 2y^6 + z^6 = 1$. Prove: except for a finite number of points this is a smooth (possibly empty) curve. Find these "special" points.

2.
 - a. Prove: every C^0 -atlas on S^n has at least two charts. (Hint: compactness)
 - b. What is the minimal number of C^∞ -charts for the torus $S^1 \times S^1$?
 - c. Prove: for any differentiable map $S^n \xrightarrow{f} \mathbb{R}^n$ there exists a point where $f'|_{x_o}$ is degenerate. Conclude: S^n is non-embeddable into \mathbb{R}^n .
 - d. Let X be a compact C^1 -manifold of dimension n . Prove: X is non-embeddable into \mathbb{R}^n .
 - e. (Assuming that $S^1 \times S^1$ is not diffeomorphic to S^2) Prove: $S^1 \times S^1$ is non-embeddable into S^2 .

3.
 - a. Take a differentiable self-map $f \circlearrowleft (\mathbb{R}^n, o)$. Prove: $f'|_o = Id$ iff $f(x) - x = o(x)$.
 - b. Suppose a subset $X \subset \mathbb{R}^n$ is defined by the equation $f(\underline{x}) = 0$. Suppose $f'|_{x_o} \neq 0$ at a point $x_o \in X$. Prove: the tangent plane to X at x_o is defined by the equation $f'|_{x_o}(\underline{x} - \underline{x}_o) = 0$. Extend this to the case of several equations.
 - c. For a manifold-germ $(X, o) \subset (\mathbb{R}^N, o)$ and a parameterized curve germ $(\mathbb{R}^1, o) \ni t \rightarrow x(t) \in (X, o)$ take the velocity $\frac{dx(t)}{dt}|_{t=o} \in T_{(\mathbb{R}^N, o)}$. Prove: $T_{(X, o)}$ is the union of such velocity vectors over all the curve germs.
 - d. Let $X = \{f_1(x) = 0 = \dots = f_k(x)\} \subset \mathbb{R}^N$, with $\text{rank}[f'|_{x_o}] = k$. Prove: $T_{(X, x_o)} = \{v \mid f'|_{x_o}(v) = 0\} \subset T_{(\mathbb{R}^N, x_o)}$.
 - e. For $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ prove: $T_{(X \times Y, (x_o, y_o))} = T_{(X, x_o)} \times T_{(Y, y_o)}$.
 - f. Given maps of manifolds $X \xrightarrow{f} Y \xrightarrow{g} Z$, prove the chain rule: $(g \circ f)' = g' \circ f'$.

4. Take a C^2 -map of manifolds $X \xrightarrow{f} \mathbb{R}$. Verify:
 - a. The condition " $f'|_{x_o} = 0$ " is independent of the choice of charts.
 - b. What about the condition " $\det[f''|_{x_o}] = 0$ "?
 - c. The condition " $f'|_{x_o} = 0$ and $\det[f''|_{x_o}] = 0$ " is independent of the choice of charts.