Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

Homework 3. Submission date: 1.12.2024

Questions to submit: 1.a. 1.b. 2.a. 2.b.ii. 4.a. 4.b. 4.c. 5.a.

(Either typed or in readable handwriting and scanned in readable resolution.)

The space $Mat_{n\times n}(\mathbb{R})$ is considered below with the norm $\|A\| = \sqrt{trace(A\cdot A^t)}$.

1. Define the map $Mat_{n\times n}(\mathbb{R}) \overset{exp}{\to} Mat_{n\times n}(\mathbb{R})$ by $exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}$. (Convention: $A^0 = \mathbb{1}$)

- **a.** Prove: $exp(A)$ converges absolutely, and the convergence is uniform on compact subsets of $Mat_{n\times n}(\mathbb{R})$. You can use $\|A\cdot B\| \le \|A\|\cdot\|B\|$ (follows from Cauchy-Schwarz inequality).
- **b.** Deduce: exp is a C^ω-map of manifolds. (You can use the Weierstraß theorem)
- **c.** Take the (complex) Jordan form, $A = U^{-1}(Diag + Nilp)U$, where $U \in GL(n,\mathbb{C})$, while the matrices $Diag, Nilp \in Mat_{n\times n}(\mathbb{C})$ are diagonal, resp. strictly upper-triangular. Verify: $Nilp^n = \mathbb{O}$ and $Diag\cdot Nilp = Nilp\cdot Diag$.
  Prove: $exp(A) = U^{-1}\cdot exp(Diag)\cdot(\sum_{k=0}^{n}\frac{Nilp^k}{k!})\cdot U$.
- **d.** Prove: if A, B commute then $exp(A+B) = exp(A)\cdot exp(B)$.
- **e.** Fix some $A \in Mat_{n\times n}(\mathbb{R})$ and define the path $\mathbb{R}^1 \overset{\gamma}{\to} Mat_{n\times n}(\mathbb{R})$, by $\gamma(t) = exp(t\cdot A)$. Compute $\frac{d\gamma}{dt}$.
- **f.** Define the function $ln(\mathbb{1}+A)$ for $\|A\| < 1$, and establish the corresponding properties a.,…, e.
- **g.** Verify: $exp(ln(\mathbb{1}+A)) = \mathbb{1}+A$ (assuming $\|A\| < 1$) and $ln(exp(B)) = B$ (assuming $\|B-\mathbb{1}\| < 1$).

2.

- **a.** Consider the function $Mat_{n\times n}(\mathbb{R}) \overset{det}{\to} \mathbb{R}$. Identify the partial derivatives $\frac{\partial det(A)}{\partial a_{ij}}$.
  Identify the critical locus of det, and the critical values.
- **b.** Prove: the following subsets of $Mat_{n\times n}(\mathbb{R})$ are C^∞-manifolds. Determine their dimensions. Which of these are compact/path-connected? Identify the tangent spaces to these manifolds at $\mathbb{1} \in Mat_{n\times n}(\mathbb{R})$.
  i. $GL(n)$ ii. $SL(n)$ iii. $O(n)$.
  These groups are called Lie-groups, while their tangent spaces are called Lie-algebras.
- **c.** Establish C^∞-diffeomorphisms: $GL(2) \cong \mathbb{R}^4 \setminus \{x^2+y^2 = z^2+w^2\}$, $SL(2) \cong \{x^2+y^2 = 1+z^2+w^2\}$.

3.

- **a.** Prove: $TS^1 \overset{C^\infty}{\approx} S^1 \times \mathbb{R}^1$. (We did this in the class.)
- **b.** Verify the functorial properties of tangent bundle: each morphism $X \overset{f}{\to} Y$ induces $TX \overset{f_*}{\to} TY$, which satisfies the chain rule, and $(Id_X)_* = \ldots$

4.

- **a.** Establish (prove) the normal form of submersion, homework 0, q. 8.a.
- **b.** Establish (prove) the normal form of immersion, homework 0, q. 8.b.
- **c.** Take a C^2-germ, $(\mathbb{R}^n, o) \overset{f}{\to} (\mathbb{R}, o)$, with $f'|_o = 0$ and $det[f''|_o] \neq 0$. Prove:
  i. The germ of $f^{-1}(o)$ at o is path-connected.
  ii. Suppose o is a local minimum of f. There exist coordinates on $(\mathbb{R}^n, o)$ such that $f^{-1}(\epsilon)$ is a sphere for any $0 < \epsilon \ll 1$. (What happens for local maximum/saddle?)

5.

- **a.** Prove: every immersion $\mathbb{R}^1 \to \mathbb{R}^1$ is a diffeomorphism onto its image.
- **b.** Give an example of an analytic immersion $\mathbb{R}^2 \to \mathbb{R}^2$, whose fibres are either infinite or empty. (Hint: a very "simple" map $\mathbb{C} \to \mathbb{C}$)
- **c.** Suppose $X \overset{f}{\to} Y$ is injective, and each germ $(X, x_o) \overset{f}{\to} (Y, f(x_o))$ is a diffeomorphism. Prove: $X \overset{f|}{\to} f(X)$ is a diffeomorphism, with $f(X) \subseteq Y$ an open subset.
- **d.** Let $X \overset{f}{\to} Y$ be a submersion, with X compact and Y connected. Prove: f is surjective. Deduce: there can be no submersion from a compact manifold to $\mathbb{R}^n$.