# Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

**Homework 4. Submission date: 8.12.2024**

Questions to submit: 2.a. 2.c. 2.d. 3.a. 3.b. 4.a.

(Either typed or in readable handwriting and scanned in readable resolution.)

Below the deformation of a function $f_o(x) \in C^r(X)$ is a function $f_t(x) \in C^r(X \times [0,1])$.

1. a. Take a function $f \in C^r(X)$ with no zeros. Define the map $g_f : TX \to TX$ by $(x,v) \to (x, f(x)v)$. Prove: $g_f$ is a $C^r$-diffeomorphism.
   b. Prove: the natural projection $TX \to X$, $(x,v) \to x$, is a submersion.
   c. Let $(X, x_o) \xrightarrow{f} (Y, y_o)$ be a submersion. Prove: $T_{(f^{-1}(y_o), x_o)} = ker[f'|_{x_o}]$.

2. a. Prove: the function $det : Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}^1$ is Morse iff $n \leq 2$.
   b. Prove the Morse lemma. (We did this in the class.)
   c. Prove: Morse points are stable. Namely, if $f_t$ is a deformation of a Morse function $(\mathbb{R}^n, o) \xrightarrow{f_o} (\mathbb{R}^1, o)$, then there exists a neighborhood $o \in \mathcal{U} \subset \mathbb{R}^n$ such that for each $0 \leq t_o \ll 1$ the function $\mathcal{U} \xrightarrow{f_{t_o}} \mathbb{R}^1$ has exactly one critical point, which is a Morse point.
   d. Let $X \xrightarrow{f_o} \mathbb{R}^1$ be a Morse function on a compact manifold. Prove: for any deformation $f_t$ the functions $f_{t_o}$ are Morse for $t_o \ll 1$, and the number of the critical points is preserved.
   e. Show that d. fails in the non-compact case. [E.g. for $f_o : \mathbb{R}^1 \to \mathbb{R}^1$]

3. a. Let $(\mathbb{R}^1, o) \xrightarrow{f} (\mathbb{R}^1, o)$ be of order $p$, i.e. $f(x) = u(x) \cdot x^p$, where $u(x) \in C^r(\mathbb{R}^1, o)$ with $u(o) \neq 0$. Prove: a $C^r$-coordinate change on $(\mathbb{R}^1, o)$ transforms $f$ into the function $x \to (\pm)x^p$.
   b. Prove: any Morsification of $f$ from a. splits the critical point into at most $(p-1)$ Morse points.
   c. Fix a submanifold $X \subset \mathbb{R}^N$, and consider linear functions $\mathbb{R}^N \xrightarrow{l} \mathbb{R}^1$.
      i. Prove: for almost all $l \in (\mathbb{R}^N)^*$ the restriction $l|_X$ is Morse.
      ii. Let $X \subset \mathbb{R}^3$ be the torus from hwk.0, q.1.e. Prove: $l|_X$ is Morse for $l(x,y,z) \neq c \cdot z$. [A possible start: take a simple parametrization $S^1 \times S^1 \xrightarrow{f} torus \subset \mathbb{R}^3$, and consider $l \circ f$.]

4. Let $X$ be an $n$-dimensional manifold.
   a. We have constructed an embedding $X \hookrightarrow \mathbb{R}^{2n+1}$. Modify the first part of that proof to get: $X$ can be immersed into $\mathbb{R}^{2n}$.
   b. Suppose $X$ is compact. Construct a map $X \to \mathbb{R}^{2n-1}$ that is immersion outside of a finite number of points. (See the instructions in Guillemin-Pollack, exercise 11 on pg.55)

5. (The joy of bump functions)
   **a.** Take a $C^\infty$ function $\mathbb{R} \xrightarrow{\tau} \mathbb{R}_{\geq 0}$ that is flat at 0 and $\tau|_{\mathbb{R}\setminus\{0\}} > 0$. (e.g. $\tau(x) = e^{-\frac{1}{x^2}}$)
   **b.** (A bump function.) Let $f(x) := \tau(x^2 - 1)$ for $|x| \leq 1$, and 0 on $\mathbb{R}^1 \setminus [-1,1]$. Verify: $f \in C^\infty(\mathbb{R})$.
   **c.** Construct $f \in C^\infty(\mathbb{R}^n)$ satisfying: $f|_{Ball} > 0$ and $f|_{\mathbb{R}^n \setminus Ball} = 0$.
   **d.** Extend c. to an arbitrary open subset $\mathcal{U} \subset \mathbb{R}^n$. [Hint: locally finite covering of $\mathcal{U}$ by balls.] This proves a theorem of Whitney: any closed subset of $\mathbb{R}^n$ is the zero locus of a $C^\infty$ function.
   **e.** Construct a monotonic function $g \in C^\infty(\mathbb{R})$ such that $g|_{(-\infty,0]} = 0$ and $g|_{[1,\infty)} = 1$. (e.g. $g(x) := \frac{\int_0^x f(t)dt}{\int_0^1 f(t)dt}$)
   **f.** Let $K \subset \mathcal{U} \subset \mathbb{R}^n$ be a compact inside an open. Construct a $C^\infty$ function $f$ such that $f|_K = 1$ and $f|_{\mathbb{R}^n \setminus \mathcal{U}} = 0$. (Hint: if $f|_K \geq 1$ then you can take $g \circ f$, with $g$ from (c).)

   Such bump functions smoothen $\mathbb{1}_K$. They are highly useful in Analysis/Differential Geometry. For geometries over other fields (and Algebra/Arithmetics) one works hard to substitute them.