

Introduction to Differential Topology

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Homework 5. Submission date: 17.12.2024

Questions to submit: 1.b. 1.e. 2.a. 2.b. 3.a. 5.b.



(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Fix $(U, V) \in GL(m) \times GL(n)$ and define $\phi_{U,V} \circ Mat_{m \times n}(\mathbb{R})$ by $\phi_{U,V}(A) = U \cdot A \cdot V^{-1}$.
Verify: $\phi_{U,V}$ is a (linear) diffeomorphism.
b. Take the subset of matrices $\Sigma_r := \{A \mid \text{rank}(A) = r\}$. Show that $\overline{\Sigma_r} \subset Mat_{m \times n}(\mathbb{R})$ is defined by polynomial equations. Prove: $\overline{\Sigma_r} = \Sigma_r \cup \dots \cup \Sigma_0$. (For the part \supseteq use a.)
c. Deduce: the condition “Being of $\text{rank} = r$ inside the set of matrices of $\text{rank} \leq r$ ” is stable and generic.
d. Let $A = \begin{bmatrix} B_{r \times r} & C_{r \times (n-r)} \\ D_{(m-r) \times r} & E_{(m-r) \times (n-r)} \end{bmatrix}$, where $\text{rank}[B_{r \times r}] = r$. Multiply A by $\begin{bmatrix} \mathbb{I} & -B^{-1}C \\ \mathbb{O} & \mathbb{I} \end{bmatrix}$ to get:
 $\text{rank}[A] = r$ iff $E = DB^{-1}C$.
e. Prove that $\Sigma_r \subset Mat_{m \times n}$ is a submanifold, and compute $\dim \Sigma_r$.
(Hint: the multiplication of d. rectifies the germ (Σ_r, A) to a plane.)

2. a. (Dis)prove: if the intersection $X_1 \cap X_2$ is non-transversal, then $X_1 \cap X_2$ is not a manifold, or not of expected dimension.
b. (Dis)prove: if $T_{(X_1 \cap X_2, y_0)} = T_{(X_1, y_0)} \cap T_{(X_2, y_0)}$ then $(X_1, y_0) \pitchfork (X_2, y_0)$.
c. (Non-transversal intersections are pathological) Take any closed subset $C \subset \mathbb{R}^n$. Embed $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, by $\underline{x} \rightarrow (\underline{x}, 0)$. Prove: there exists a C^∞ -submanifold $X \subset \mathbb{R}^{n+1}$ satisfying $C = X \cap \mathbb{R}^n$.
(Hint: use Whitney’s theorem on zeros from homework 4.)

3. Fix submanifolds $X_1, X_2, X_3 \subset Y$ and assume $X_1 \cap X_2 \cap X_3 \neq \emptyset$. This intersection is called *transversal*, $\pitchfork X_i$, if $X_i \pitchfork (X_j \cap X_k)$ for all distinct values of i, j, k .
a. Does $\pitchfork X_i$ imply (or is implied by): $X_j \pitchfork X_i$ for all $i \neq j$?
b. Prove: $\pitchfork X_i$ implies $\text{codim}(X_1 \cap X_2 \cap X_3) = \sum_i \text{codim}(X_i)$.
c. For planes in \mathbb{R}^n prove: $\pitchfork L_i$ iff $\text{codim}(L_1 \cap L_2 \cap L_3) = \sum_i \text{codim}(L_i)$.

4. a. Suppose the C^r -maps $X \xrightarrow{f_0, f_1} Y$ are homotopic. Prove: there exists a C^r -homotopy $X \times [0, 1] \xrightarrow{\{f_t\}} Y$ satisfying: $f_t = f_0$ for $t \leq \frac{1}{4}$ and $f_t = f_1$ for $t \geq \frac{3}{4}$.
b. Prove: homotopy defines equivalence relations on $Maps(X, Y)$.
c. A set X is called contractible if the map Id_X is homotopic to a constant map. Prove:
i. X is contractible iff all the maps $Y \rightarrow X$ are homotopic for every Y .
ii. X is contractible iff all the maps $X \rightarrow Y$ are homotopic for every Y .

5. a. Prove: every connected manifold is path-connected. [For any $x_o \in X$ take its path-connected component, $x_o \in X_o \subseteq X$. Prove: X_o is both open and closed inside X .]
b. Take the map $\mathbb{R}^1 \xrightarrow{f_0} \mathbb{R}^1$, $f_0(x) = x$, which is injective, surjective, immersive and submersive. Give a polynomial deformation $f_t(x) = f_0(x) + t \cdot h(x)$ that destroys all these properties (for each $t \neq 0$). Give an analytic deformation that also destroys the properness of f_0 .
c. Consider $Maps(X, Y)$ with X compact. Prove: the properties of submersion and surjective submersion are stable. Show by example that the property of surjectivity is not necessarily stable.