Introduction to Differential Topology

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Homework 6. Submission date: 24.12.2024

Questions to submit: 1.a. 1.c. 2.c. 2.e. 3.a. 3.e.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Let $(X, x_o) = \{f_{n+1}(x) = \cdots = f_N(x) = 0\} \subset (\mathbb{R}^N, x_o)$, with $rank[f'|_{x_o}] = N - n$. Verify: $\mathcal{N}_{(X,x_o)} = Span_{\mathbb{R}}[\nabla f_{n+1}|_{x_o}, \ldots, \nabla f_N|_{x_o}] \subset \mathbb{R}^N$.

b. Verify: the map $X \xrightarrow{i} \mathcal{N}(X)$, $x \to (x, o)$, is a proper embedding. Verify: the map $\mathcal{N}(X) \xrightarrow{\pi} X$, $(x, v) \to x$, is a submersion.

c. Suppose a map $Y_1 \supset X_1 \xrightarrow{f} X_2 \subset Y_2$ extends to a map of open neighborhoods $Y_1 \supset \mathcal{U}(X_1) \xrightarrow{\tilde{f}} \mathcal{U}(X_2) \subset Y_2$. Prove: $\tilde{f}$ induces the map $\mathcal{N}(X_1/Y_1) \xrightarrow{f_*} \mathcal{N}(X_2/Y_2)$ that acts linearly on fibres and satisfies: $\pi \circ f_* \circ i = f$. Moreover, this correspondence satisfies:
i. $(f \circ g)_* = f_* \circ g_*$. ii. If $\tilde{f}$ is a diffeomorphism then so is f_*.

2. Below $X \subset \mathbb{R}^N$ is a closed subset that is a C^r-submanifold. We construct tubular neighborhoods.

a. Denote by $L^\perp_x$ the plane orthogonal to $X \subset \mathbb{R}^N$ at $x \in X$. (Thus $dim L^\perp_x = codim(X, x)$.) Prove: for $x \neq x'$ the intersections $L^\perp_x \cap L^\perp_{x'}$ (if non-empty) "do not approach X". Namely, there exists a function $X \xrightarrow{\epsilon} \mathbb{R}_{>0}$ satisfying: $Ball_{\epsilon(x_o)} \cap L^\perp_x \cap L^\perp_{x'} = \varnothing$ for each x_o and all $x \neq x'$.

b. For each $x \in X$ take the orthogonal disc $Disc^\perp_{x,\epsilon} \subset L_x$. Prove: there exists a C^r-function $X \xrightarrow{\epsilon} \mathbb{R}_{>0}$ satisfying: $Disc^\perp_{x,\epsilon(x)} \cap Disc^\perp_{x',\epsilon(x')} = \varnothing$ for all $x \neq x' \in X$.

c. Show that a. and b. do not hold for non-smooth subsets of $\mathbb{R}^N$.

d. Denote $\mathcal{U}_\epsilon(X) := \coprod_{x \in X} Disc^\perp_{x,\epsilon(x)}$. Prove: $X \subset \mathcal{U}_\epsilon(X) \subset \mathbb{R}^N$ is an open neighborhood. [For each $y_o \in Disc^\perp_{x,\epsilon(x)}$ any point of a small $Ball_{y_o,\delta}$ is at distance$< \epsilon(x)$ from X.]

e. Define the map $\mathcal{N}(X) \xhookrightarrow{\psi} \mathbb{R}^N$ by $(x, v) \to x + \frac{\epsilon(x) \cdot v}{\sqrt{1+\|v\|^2}}$. Prove: it is a proper bijection $\mathcal{N}(X) \rightleftarrows \mathcal{U}_\epsilon(X)$. Prove: $\psi'|_{(x,v)}$ is non-degenerate for each $(x, v) \in \mathcal{N}(X)$. Thus ψ is a C^r-diffeomorphism.

f. Define the map $\mathcal{U}_\epsilon(X) \to X$ by $\pi : Disc^\perp_{x,\epsilon(x)} \to x$. Factorize this map into $\mathcal{U}_\epsilon \xrightarrow{\sim} \mathcal{N}(X) \to X$. Deduce: π is a C^r-submersion.

3. a. Let $X \subset \mathbb{R}^N$ a submanifold with boundary. Prove: $\partial X \subset \mathbb{R}^N$ is a submanifold without boundary.

b. (Dis)prove: $X \supset \partial X$ is closed and $\mathbb{R}^N \supset \partial X$ is closed.

c. (Dis)prove: $X \supset \partial X$ is compact iff $\mathbb{R}^N \supset \partial X$ is compact.

d. Let $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ submanifolds with boundaries. Prove: $X \times Y \subset \mathbb{R}^N \times \mathbb{R}^M$ is a submanifold with boundary iff $[\partial X = \varnothing$ or $\partial Y = \varnothing]$.

e. Let $\mathbb{R}^N \supset X \xrightarrow[\sim]{f} Y \subset \mathbb{R}^M$ be a diffeomorphism of subsets. Prove: if X is a submanifold with boundary, then so is Y, and moreover: $f_| : \partial X \xrightarrow{\sim} \partial Y$.

f. Deduce: Möbius$\not\approx \mathbb{R}^1 \times S^1$.

g. (Local rectification) Prove: $\forall\, x_o \in \partial X$ there exists a diffeomorphism $(\mathbb{R}^N, x_o) \times \mathbb{R}^N_v \xrightarrow{\psi} (\mathbb{R}^N, o) \times \mathbb{R}^N_v$, linear on $\mathbb{R}^N_v$, such that $\psi(T(X, x_o)) = (H, o) \times \mathbb{R}^N_v \subset (\mathbb{R}^n, o) \times \mathbb{R}^N_v \subset (\mathbb{R}^N, o) \times \mathbb{R}^N_v$.

h. (functoriality) Given a C^r-map $X \xrightarrow{f} Y$, define $TX \xrightarrow{f_*} TY$ by $(x, v) \to (f(x), f'|_x(v))$.

i. Verify: f_* is a C^r-map, and it is linear on the fibres of TX.

ii. Prove: $(f \circ g)_* = f_* \circ g_*$, and if f is invertible, then so is f_*.