Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

Homework 7. Submission date: 31.12.2024

Questions to submit: 1.c. 2.a. 2.d. 2.e. 3.a. 3.b. 3.d.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Verify: a map $X \xrightarrow{f} Y$ is transversal to $y \in Y$ iff y is a regular value of f and of $f|_{\partial X}$.

b. Consider maps $X \to Y \supset Z$. Prove: if a subset $Z \subset Y$ is not closed, then the transversality to Z is not a stable property. (I.e. exists a map $f_o \pitchfork Z$, and a deformation f_s, such that …)

c. Let $X \xrightarrow{f} Y \supset Z$ with $f \pitchfork Z$. Prove: $T(f^{-1}Z) = (f')^{-1}(TZ)$.

d. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \supset W$, with $\partial Y, \partial Z, \partial W = \varnothing$. Suppose $g \pitchfork W$. Verify: $f \pitchfork g^{-1}(W)$ iff $(g \circ f) \pitchfork W$. Verify: $I_2(f, g^{-1}(W)) = I_2(g \circ f, W)$.

e. Let $X \xrightarrow{f} \mathbb{R}$, with $\partial X = \varnothing$ and 0 a regular value. Prove: $f^{-1}(\mathbb{R}_{\geq 0}) \subset X$ is a manifold with boundary and $\partial(f^{-1}\mathbb{R}_{\geq 0}) = f^{-1}(0)$.

2. Fix a submanifold $X \subset Y$, with $\partial X = \varnothing = \partial Y$. A section of the normal bundle $\mathcal{N}({}^{X}\!/_{Y}) \xrightarrow{\pi} X$ is a map $X \xrightarrow{s} \mathcal{N}({}^{X}\!/_{Y})$ satisfying $\pi \circ s = Id_X$.

The normal bundle is called *trivial* if there exists a diffeomorphism $\mathcal{N}({}^{X}\!/_{Y}) \xrightarrow{\sim} X \times \mathbb{R}^k$ whose restrictions to the fibres are linear isomorphisms.

a. Verify: the normal bundle is always locally trivial. (For each $x \in X$ exists a neighborhood $x \in \mathcal{U} \subset X$ such that…)

b. Verify: the set of sections of $\mathcal{N}(X/Y)$ is a vector space.

c. Verify: any section $s : X \to \mathcal{N}(X/Y)$ is an embedding.

d. Let $X \subset \{f(x) = 0\} \subset \mathbb{R}^N$. Verify: the map $x \to f'|_x$ defines a section of $\mathcal{N}(X)$.

e. Let $S^{n-1} = \{x_{n+1} = 0\} \subset S^n \subset \mathbb{R}^{n+1}$. Prove: $\mathcal{N}(S^{n-1}/S^n)$ is trivial.

f. Denote $k = dimY - dimX$. Prove: $\mathcal{N}(X/Y)$ is trivial iff there exist sections $s_1 \dots s_k$ such that the vectors $s_1|_{x_o}, \dots, s_k|_{x_o}$ are linearly independent for each point $x_o \in X$.

g. Prove: $\mathcal{N}(X/Y)$ is trivial iff there exists an open neighborhood $X \subset \mathcal{U}(X) \subset Y$ and a presentation $X = \{f_1(x) = 0 = \dots = f_k(x)\} \subset \mathcal{U}(X)$, such that $rank[\underline{f}'|_x] = k$ for each $x \in X$. (Hint: the tubular neighborhood)

h. Let $S^1 \hookrightarrow$Möbius be the equatorial embedding ("at height 0"). Prove: $\mathcal{N}(S^1/M\ddot{o}b)$ is non-trivial.

3. a. X is called contractible if the map $Id_X \circlearrowright X$ is homotopic to a map $X \to x_o \in X$. Prove: if X or Y or Z are contractible then $I_2(f, Z) = 0$. Here $0 < dim(Z) < dim(Y)$ and $f(X) \subsetneq Y$.

[The intersection theory is "empty" in the contractible case.]

b. Suppose a compact manifold X (with $\partial X = \varnothing$) is contractible. Prove: X=one point. [Hint: $deg_2 Id_X$]

c. Prove: any map $S^1 \xrightarrow{f} S^1$ is homotopic to $z \to z^p$.

E.g.: start from the coverings $[0,1] \xrightarrow{\sigma} S^1 \xrightarrow{f} S^1 \leftarrow \mathbb{R}^1$, and lift f to $\tilde{f} := log \circ f \circ \sigma$, using branches of log. One can assume: $\tilde{f}(0) = 0$. Take the homotopy $\tilde{f}_s(t) = s \cdot t \cdot \tilde{f}(t) + (1 - s) \cdot \tilde{f}(t)$. Verify that it descends to S^1.

d. Prove: the manifolds S^2, $S^1 \times S^1$, $Klein$ are pairwise non-diffeomorphic. (E.g.: compute $I_2(C_1, C_2)$ for various S^1-embeddings.)