Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

Homework 8. Submission date: 13.01.2025

Questions to submit: 1.b. 2.d. 2.e. 3.b. 4.a. 5.b.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Let X be compact, $\partial X = \varnothing$, and Y is connected. Suppose $X \xrightarrow{f} Y$ has $deg_2(f) \neq 0$. Prove: f is surjective.
In particular, if Y is non-compact, then necessarily $deg_2(f) = 0$.

b. Let $f \pitchfork Z$, with $dim(X) + dim(Z) = dim(Y)$. Prove:

i. If Z is not closed, then $I_2(f, Z)$ is not a homotopy invariant.

ii. If X is not compact, then $I_2(f, Z)$ is not a homotopy invariant.

iii. If $\partial X \neq \varnothing$, then $I_2(f, Z)$ is not a homotopy invariant.

2. a. Let $(V_1, \mathcal{B}_1)$, $(V_2, \mathcal{B}_2)$ be vector spaces with bases. Verify: the orientations of $(V_1 \oplus V_2, \{\mathcal{B}_1, \mathcal{B}_2\})$ and of $(V_1 \oplus V_2, \{\mathcal{B}_2, \mathcal{B}_1\})$ differ by $(-1)^{dimV_1 \cdot dimV_2}$.

b. Prove: any orientable connected manifold has exactly two orientations.
What happens if X has r connected components?

c. Given $X \subset Y$, with $dim(X) = dim(Y)$ and X non-orientable. Prove: Y is non-orientable.
Deduce: the Klein bottle and the real projective plane ($\mathbb{RP}^2$) are non-orientable.

d. Cut-out the "equator" $S^1 \subset$Möbius. Verify: Möbius$\setminus S^1 \cong S^1 \times \mathbb{R}^1$. (In particular orientable.)

e. Cut-out the "equators" $S^1 \subset$Klein, $\mathbb{RP}^2$. Identify the obtained surfaces.

3. a. For $X \subset Y$ verify: $TY|_X = TX \oplus \mathcal{N}({}^{X}\!/_{Y})$. Verify: fixing orientations of any two of these bundles determines the orientation of the third.

b. Take a hypersurface $X = \{f(x) = 0\} \subset \mathbb{R}^{n+1}$, suppose f' has no zeros on X. Prove: X is orientable.

c. Deduce: none of Möbius, Klein, $\mathbb{RP}^2$ can be presented as $X = \{f(x) = 0\} \subset Y$, where Y is orientable of $dim = 3$, and f' has no zeros on X.

d. Suppose X is simply connected, i.e. all its loops are contractible. (I.e. every map $S^1 \to X$ is homotopic to a constant map.) Prove: X is orientable.

e. Take a manifold $X = \mathcal{U}_1 \cup \mathcal{U}_2$, with $\mathcal{U}_1, \mathcal{U}_2$ open and orientable. Suppose $\mathcal{U}_1 \cap \mathcal{U}_2$ is path-connected. Prove: X is orientable.

f. Let $X \subset Y$, with $dim(X) + 1 = dim(Y)$, and Y-orientable. Prove: X is orientable iff $\mathcal{N}({}^{X}\!/_{Y})$ is trivial. [In the sense of q.2. of hwk.7.]

4. a. Take $H_n := \{x_n \geq 0\} \subset \mathbb{R}^n$. Is the orientation of ∂H_n (as the boundary of H_n) compatible with the standard orientation of $\mathbb{R}^{n-1}_{x_1 \dots x_{n-1}}$?

b. Prove: for any manifold X the bundle $\mathcal{N}({}^{\partial X}\!/_{X})$ is orientable.

c. Prove: ∂X is orientable iff there is an orientable neighborhood $\partial X \subset \mathcal{U}(X) \subset X$.

5. a. Fix some orientations $\omega(X), \omega(Y)$, where at least one of these manifolds has no boundary. For each point $(x, y) \in X \times Y$ define the orientation of $T_{(X \times Y, x \times y)}$ via $\{\mathcal{B}_X, \mathcal{B}_Y\}$. Verify: this defines orientation on $X \times Y$.

b. Take an embedding $S^1 \overset{\phi_o}{\hookrightarrow} X$ into a smooth surface. Prove: if X is orientable, then there exists deformation of ϕ_o satisfying: $\phi_o(S^1) \cap \phi_\epsilon(S^1) = \varnothing$ for $\epsilon \neq 0$.

c. Deduce (once again): Möbius, Klein, $\mathbb{RP}^2$ are non-orientable.

d. Prove: any compact connected submanifold in $\mathbb{R}^{n+1}$ with no boundary is orientable. (You can use the Jordan-Brouwer separation theorem.)

e. Deduce: Klein and $\mathbb{RP}^2$ are non-embeddable into $\mathbb{R}^3$. (Did we prove this already in 3.c?)