# Introduction to Differential Topology

BGU, 201.2.7061, Fall 2024, D. Kerner

**Homework 9. Submission date: 24.01.2025**

Questions to submit: 1.b. 2.b. 4.b. 4.d. 4.g. 4.i. 5.b.

(Either typed or in readable handwriting and scanned in readable resolution.)

**1.** Take $Ball_1(o) \subset \mathbb{R}^n$, with the standard orientation.

**a.** Take $S^{n-1} = \partial Ball$, with the induced orientation. At the north pole identify $T_{(S^{n-1},\hat{x}_n)}$ with $\mathbb{R}^{n-1}_{x_1\dots x_{n-1}}$. Do we get the standard orientation?

**b.** Define $\mathbb{R}^n \xrightarrow{\rho} \mathbb{R}_{\geq 0}$ by $\rho(x) = \|x\|^2$. Verify: $\rho \pitchfork \{1\}$. Is the induced orientation on the sphere $\rho^{-1}(1)$ the same as in a.?

**c.** Let $X$ be connected, orientable, $\partial X = \varnothing$. Prove: both orientations on $X$ induce the same orientation on $X \times X$.

**d.** Let $X$ be non-orientable. Prove: $X \times Y$ is non-orientable for any $Y$. [Hint: start from $Y = \mathbb{R}^n$.]

**2. a.** Prove: $I(f, \omega_X, Z) = -I(f, -\omega_X, Z)$.

**b.** Let $X_1, X_2 \subset Y$ with $dimX_1 + dimX_2 = dimY$. Prove: $I(X_1, X_2) = (-1)^{dim(X_1)\cdot dim(X_2)} I(X_2, X_1)$. For $dimX_1 + dimX_2 \geq dimY$ prove: $\omega(X_1 \cap X_2) = (-1)^{codim(X_1)\cdot codim(X_2)} \omega(X_2 \cap X_1)$.

**c.** [Chain rule for $I(*,*)$] Given $\tilde{X} \xrightarrow{f} X \xrightarrow{g} Y \supset Z$. Suppose they are all appropriate for the intersection theory. Prove: $I(g \circ f, Z) = I((g, f^{-1}(Z))$.

**3.** [The boundary issue] Let $\partial X \subset X \xrightarrow{f} Y \supset Z$, with $X, Z$-connected and $f \pitchfork Z$.

- Take the induced orientation on $f^{-1}(Z)$. It gives the orientation $\omega[\partial(f^{-1}(Z))]$. (Outer normal)
- Present: $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X = f|^{-1}_{\partial X}(Z)$. Thus we have also the orientation $\omega[f|^{-1}_{\partial X}(Z)]$.

Prove: $\omega[\partial(f^{-1}(Z))] = (-1)^{codim(Z)} \cdot \omega[f|^{-1}_{\partial X}(Z)]$.

**4. a.** Compute the degree of the antipodal map $S^n \xrightarrow{-Id} S^n$, $x \to -x$.

**b.** Prove: $Id_{S^n}$ is homotopic to $-Id_{S^n}$ iff $n$ is odd.

**c.** Prove: there exists a vector field with no zeros on $S^n$ iff $n$ is odd. [We did this in the class.]

**d.** Let $(\mathbb{C}, o) \xrightarrow{f(z)} (\mathbb{C}, o)$ be an analytic isomorphism. Prove: $deg_o f = +1$.

**e.** Compute the degree of the anti-holomorphic map $(\mathbb{C}, o) \to (\mathbb{C}, o)$ $z \to \bar{z}^d$.

**f.*** (If you've seen $\mathcal{O}$-functions of several variables) Extend d. to the case $(\mathbb{C}^n, o) \xrightarrow{f(z)} (\mathbb{C}^n, o)$.

**g.** Let $X \xrightarrow{f} Y$, both connected, compact, oriented, $\partial X = \varnothing = \partial Y$. Prove: if $degf \neq 0$ then $f$ is surjective.

**h.** Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function that extends to a continuous function $\tilde{f} : S^2 \to S^2$. (Thus $f$ has either a pole at $|z| = \infty$ or a finite limit.) Prove: either $f = const$ or $\tilde{f}$ is surjective.

**i.** Compute the degree of the map $S^1 \to S^1$ defined by $z \to \frac{h(z)}{|h(z)|}$, where $h(z) = 10z^p - e^{re[z]} - i \cdot \sqrt{2 - im[z]}$.

**j.** (Rouché-type lemma) Take $C^1$-maps $\mathbb{R}^2 \supset \overline{\mathcal{U}}_{compact} \xrightarrow{f, f+g} \mathbb{R}^2$. Take their zeros, $f^{-1}(o) = \{x_j\}$ and $(f+g)^{-1}(o) = \{\tilde{x}_j\}$. Suppose $\|f|_{\partial\mathcal{U}}\| > \|g|_{\partial\mathcal{U}}\|$. Then $\sum deg_{x_j} f = \sum deg_{\tilde{x}_j}(f+g)$.

**k.** (Chain rule for $deg$) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, prove: $deg[g \circ f] = deg[g] \cdot deg[f]$.

**5. a.** Prove: any map $S^1 \to S^1 \times S^1$ is homotopic to $z \xrightarrow{f_{p,q}} (z^p, z^q)$. [Use q.3.c of hwk.7]

**b.** Orient $S^1$ counterclockwise, take the product orientation $\omega(S^1 \times S^1) = \omega(S^1) \times \omega(S^1)$.

Prove: $I(f_{p,q}(S^1), f_{p',q'}(S^1)) = [p', q'] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$. Deduce: $Maps(S^1, S^1 \times S^1)/_{homotopy} \cong \mathbb{Z} \oplus \mathbb{Z}$.