Geometric Calculus 2, 201.1.1041

Moed.A, 20.07.2025, three hours.

(Lecturer: Dmitry Kerner)

No auxiliary material is allowed.

Solve all the questions. The total is 105 points.

Do not write in red color!

1. (10 points) Take a path-connected manifold X. Prove: the dimension $dim(X, x_o)$ does not depend on the point $x_o \in X$.

2. (10 points) Take the reflection map $\mathbb{R}^n \ni x \to -x \in \mathbb{R}^n$. Does it preserve the orientation of $\mathbb{R}^n$?

3. (15 points) Define the function $\mathbb{R}^1 \supset (0,1) \stackrel{x}{\to} \mathbb{R}^5$ by $x(t) = (e^t, e^{-t}, t, cos(t), sin(t))$.
a. Prove: its image is a bounded smooth curve.
b. Compute its length.

4. (15 points) Take the subset $X = \{\sum x_i^6 = 1 - \sum x_i^2\} \subset \mathbb{R}^n$. Prove: X is a smooth compact orientable manifold.

5. (20 points) Take the polar coordinates map of the standard sphere $(0, 2\pi)_\phi \times (0, \pi)_\theta \stackrel{f}{\to} S^2 \subset \mathbb{R}^3$. Take the forms $\omega = xdx + ydy - zdz \in \Omega^1(\mathbb{R}^3)$ and $\Omega = dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)$.
Write the explicit expressions for $f^*(\omega|_{S^2})$ and $f^*(\Omega|_{S^2})$ in the coordinates (ϕ, θ).

6. (15 points) Take a C^r-manifold with some coordinate charts $X = \cup \mathcal{U}_\alpha$. Prove: each form $\omega \in \Omega^k(X)$ is presentable as $\sum_{\alpha,i} C_{\alpha,i} dg_{\alpha,i,1} \wedge \cdots \wedge dg_{\alpha,i,k}$, where $C_{\alpha,i}, g_{\alpha,i,j} \in C^{r-1}(X)$ and $supp(C_{\alpha,i}) \subseteq \mathcal{U}_\alpha$.

7. (20 points) Take the standard sphere $S^2 \subset \mathbb{R}^3$ with the orientation corresponding to the inner normal. Let $\omega = z \cdot x \cdot dz \wedge dy|_{S^2} \in \Omega^2(S^2)$. Compute $\int_{S^2_+} \omega$, where S^2_+ is the upper hemisphere.

Good Luck!