# Geometric Calculus 2

201.1.1041 Spring 2025 (Dmitry Kerner)

## Homework 0. Not for submission

*Our course is heavily based on the previous 3 semesters.*
*It is important to (fully) solve this homework before the first lecture.*

Notations&conventions:

- The unit vector in $j$'th direction is $\hat{x}_j \in \mathbb{R}^n$. A point in the standard coordinates is $\underline{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$.
- An open subset $\mathcal{U}\subseteq\mathbb{R}^n$. The standard sphere $\mathbb{S}^{n-1}:=\{\underline{x}|\ \|\underline{x}\|=1\}\subset\mathbb{R}^n$.
- The partial derivatives $\partial_1 f,\dots,\partial_n f$. The (total) derivative of order $k$ at a point $f^{(k)}|_{x_o}$. Thus $f^{(1)}|_{x_o}$ is a vector, $f^{(2)}|_{x_o}$ is a (symmetric) matrix, and so on.
- Denote by $C^k(\mathcal{U})$ the ring of functions with continuous $k$'th derivative. (Here $1\leq k\leq\infty$)

**1.** a. Prove: the series $\sum a_j x^j$ converges on $(-r,r)$ iff $|a_j|<\frac{C}{r^j}$ for some $C\in\mathbb{R}$ (and all $j\in\mathbb{N}$).

b. Suppose the series $\sum a_j x^j$ converges on $(-r,r)$. Prove: $\frac{d}{dx}(\sum a_j x^j)=\sum\frac{d}{dx}(a_j x^j)$, and the new series also converges on $(-r,r)$.

c. Suppose a function $(-\epsilon,\epsilon)\xrightarrow{f}\mathbb{R}$ is infinitely differentiable. Suppose $|f^{(j)}|_x|<\frac{C}{r^j}$, for a fixed $C$, and all $j\in\mathbb{N}$ and all $x\in(-\epsilon,\epsilon)$. Prove: $f=Taylor_o[f]$. Such a function is called "analytic at $o$."

d. Verify: $f=Taylor_o[f]$ for the functions $e^x$, $(1+x)^\alpha$ (for $\alpha>0$), $ln(1+x)$, $\frac{1}{1+x^2}$.
In each case specify the largest interval on which the equality holds.

e. For $\mathcal{U}\subset\mathbb{R}$ define $C^\omega(\mathcal{U}):=\{$functions analytic at all the points of $\mathcal{U}\}$. Namely, for each $x_o\in\mathcal{U}$ one has: $f=Taylor_{x_o}[f]$ locally near $x_o$.
Prove: $C^\omega(\mathcal{U})$ is a commutative ring.

f. Verify: $f(x)=e^{-\frac{1}{x^2}}\in C^\infty(\mathbb{R}^1)$, but $f\neq Taylor_o[f]$. (For any neighborhood $o\in(-\epsilon,\epsilon)\subset\mathbb{R}^1$.)

**2.** a. Let $A\in Mat_{m\times n}(\mathbb{R})$ be of the maximal rank. For $m\leq n$ prove: there exists $U\in GL(n,\mathbb{R})$ satisfying: $A\cdot U=[\mathbb{1}|\mathbb{O}]$.
What is the corresponding statement for $m\geq n$?

b. Let $A\in Mat_{n\times n}(\mathbb{R})$ and suppose $v_1,\dots,v_k$ are eigenvectors with pairwise distinct eigenvalues $\lambda_1,\dots,\lambda_k$. Prove: $v_1,\dots,v_k$ are linearly independent.

c. Prove: $\det[\mathbb{1}+t\cdot A]=1+t\cdot trace(A)+O(t^2)$.

d. Let $A=\{a_{ij}(t)\}\in Mat_{n\times n}$, here $a_{ij}(t)$ are differentiable functions of one variable. Prove:

$$det[A]'=det\begin{bmatrix}--a_{1\bullet}(t)'--\\--a_{2\bullet}(t)--\\\dots\end{bmatrix}+det\begin{bmatrix}--a_{1\bullet}(t)--\\--a_{2\bullet}(t)'--\\--a_{3\bullet}(t)--\\\dots\end{bmatrix}+\dots+det\begin{bmatrix}--a_{1\bullet}(t)--\\\dots\\--a_{n\bullet}(t)'--\end{bmatrix}.$$

**3.** Draw/describe the following subsets of $\mathbb{R}^3$: i. $\left\{1-x^2-y^2\geq z\geq x^2+y^2-1\right\}$

ii. $\left\{0\leq z\leq\frac{1}{xy},\ |x|+|y|\leq 1\right\}$ iii. $\left\{x^2+y^2+z^2\leq 1,\ (x-\frac{1}{2})^2+(y-\frac{1}{2})^2\leq 1\right\}$

iv. $\left\{\sqrt{x^2+y^2}\leq z\leq\sqrt{1-x^2-y^2}\right\}$ v. $\left\{z^2\leq x^2+y^2+1,\ x^2+y^2\leq 1\right\}$.

Are they open/closed/path-connected? Describe their boundary. Is the boundary a smooth surface?

**4.** a. Make the full $(2\pi)$ rotation of the curve $\{(y-r)^2+z^2=R^2,\ x=0\}\subset\mathbb{R}^3$ around the $z$ -axis. Here $0<r<R$. Draw this surface. (It is called: a torus) Write the defining equation.

b. Define the map $S^1_\phi \times S^1_\psi \xrightarrow{f} \mathbb{R}^3_{xyz}$ by $f_x(\phi,\psi)=(R+r\cdot sin(\phi))cos(\psi)$, $f_y(\phi,\psi)=(R+r\cdot sin(\phi))sin(\psi)$, $f_z(\phi,\psi)=r\cdot cos(\phi)$. Here $S^1_\phi$ is parameterized by $(sin(\phi), cos(\phi))$.
  - i. Verify: $\underline{f}$ is the parametrization of this torus. [I.e. $f$ is bijective onto its image, and the matrix $[f']_{3\times 2}$ is of constant rank 2.]
  - ii. Find and classify the critical points of $f_x$. [What is their geometric meaning?]
  - iii. What are the critical points of $f_z$?

c. A subset $X \subset \mathbb{R}^N$ is defined locally near a point $x_o \in X$ by the equation $f(x)=0$. Suppose $f'|_{x_o} \neq 0$. Prove: the tangent plane to $X$ at $x_o$ is defined by the equation $(x-x_o)\cdot \nabla f|_{x_o}=0$.

d. Let $S^1 \times S^1 = \{\underline{x} |\ x_1^2+x_2^2=1=x_3^2+x_4^2\} \subset \mathbb{R}^4$.
  - i. Find the equations for the tangent plane to this subset at a point $\underline{x}_0$.
  - ii. Check that none of these planes intersects the subset $\Big(Ball_1(0,0)\times\mathbb{R}^2\Big) \cup \Big(\mathbb{R}^2 \times Ball_1(0,0)\Big)$.

**5.** Define the polar coordinates in $\mathbb{R}^n$ by $r=\sqrt{\sum x_i^2}$, $\Big\{\phi_j = arccos\frac{x_j}{\sqrt{\sum_{i=1}^j x_i^2}} \in [0,\pi]\Big\}_{j=3,\dots,n}$, while $\phi_2 \in [0,2\pi)$ is determined by $arccos\frac{x_1}{\sqrt{x_1^2+x_2^2}}$ and $arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}$.

a. Verify: for $n=2,3$ we get the ordinary polar coordinates in $\mathbb{R}^2$, $\mathbb{R}^3$.

b. Why do we need two expressions for the angle $\phi_2$?

c. Give the geometric definition of all the angles $\phi_2,\dots,\phi_n$. [E.g. $\phi_n$ is the angle with $\hat{x}_n$]

d. Write the explicit formulas for the map $(r,\underline{\phi}) \to \underline{x}$.

e. Verify: the domain of definition of the coordinate change $\underline{x} \to (r,\underline{\phi})$ is $\mathbb{R}^n \setminus \{x_1^2+x_2^2=0\}$, while it image is $\mathbb{R}_{>0}\times(0,\pi)^{n-2}\times[0,2\pi)$.

f. Verify: the map $\mathbb{R}_{>0}\times(0,\pi)^{n-2}\times[0,2\pi) \ni (r,\underline{\phi}) \to \underline{x} \in \mathbb{R}^n\setminus\{x_1^2+x_2^2=0\}$ is $C^\infty$ and bijective. Is its inverse $C^\infty$ as well?

g. Using polar coordinates prove: the sphere $S^{n-1}\subset\mathbb{R}^n$ is compact and path-connected.

**6.** a. Given $\mathbb{R}^n \supseteq \mathcal{U} \xrightarrow{f} \mathbb{R}$ and $v\in\mathbb{R}^n$, the directional derivative is defined by $\partial_v f|_{x_o} := lim_{t\to 0}\frac{f(x_o+t\cdot v)-f(x_o)}{t}$. For $f\in C^1(\mathcal{U})$ prove: $\partial_v f|_{x_o} = \nabla f\cdot v$.

b. The notions of min/max/saddle of $\mathcal{U}\xrightarrow{f}\mathbb{R}$ are defined without fixing the coordinates. Verify: the conditions "$f'|_{x_0}=0$, $f''|_{x_0}$ is positive/negative definite" are preserved under local $C^2$-coordinate changes.

c. Suppose a level curve of $f(x,y)\in C^1(\mathbb{R}^2)$ is defined by $y^2=x^2$. Prove: $f'|_{(0,0)}=0$.
(Note: the prescribed level curve does not imply $f(x,y)=y^2-x^2$. E.g. all $f(x)=x^\alpha$ have the same zero level set for $\alpha>0$.)

**7.** a. Give an example of a function $\mathbb{R}\xrightarrow{f}\mathbb{R}$ such that $f'|_0=0$, but $f$ is $C^0$-globally invertible.

b. Prove: if $\mathbb{R}\xrightarrow{f}\mathbb{R}$ is $C^1$ and $f'|_0=0$ then $f$ is not $C^1$-locally invertible at $x=0$.

c. Write the Jacobian matrix for the coordinate change to polar coordinates, $(x,y)\to(r,\phi)$.

d. Take a $C^r$-map $\mathbb{R}^2 \supset Ball_\epsilon(o) \xrightarrow{\phi} \mathbb{R}^2$ of the form $(x_1,x_2)\to(x_1+h_1(\underline{x}), x_2+h_2(\underline{x}))$, where $\underline{h}(\underline{x})\in O(\|\underline{x}\|^2)$. Prove: $\phi$ is invertible locally near the origin, and $\phi^{-1}\in C^r$. (Thus $\phi$ is a local $C^r$-coordinate change.)

e. Take a $C^r$-function $\mathbb{R}^n\xrightarrow{f}\mathbb{R}^m$ with $r\geq 1$. Suppose $f(o)=o$. Establish the normal form:
  - i. If $n\geq m$ and $rank[f'|_o]=m$, then in some local ($C^r$) coordinates at $o\in\mathbb{R}^m$ the function is: $f(\underline{x})=(x_1,\dots,x_m)$.
  - ii. If $n\leq m$ and $rank[f'|_o]=n$, then in some local ($C^r$) coordinates at $o\in\mathbb{R}^n$ the function is: $f(\underline{x})=(x_1,\dots,x_n,0,\dots,0)$.

f. Open mapping theorem: if $\mathbb{R}^m\supseteq\mathcal{U}_f\xrightarrow{f}\mathbb{R}^n$ is $C^1$, $n\leq m$, and $rank[f']=n$ everywhere on $\mathcal{U}_f$, then $f$ sends open sets to open sets. Give different proofs (via the implicit function theorem, via the inverse function theorem, via the normal form)