Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 1.

Submission date: 30.03.2025.

Questions to submit: 1.b. 1.c. 1.f. 2.a.i. 3.e

1. Define the map $Mat_{n\times n}(\mathbb{R}) \stackrel{\phi}{\to} \mathbb{R}^{n^2}$ by $\phi(A) = \{a_{ij}\}$ (the long vector of all the matrix entries). Define the norm on $Mat_{n\times n}(\mathbb{R})$ by $\|A\| = \sqrt{trace(A\cdot A^t)}$. Verify: this norm is induced from the standard norm on $\mathbb{R}^{n^2}$, i.e. $\|A\| = \|\phi(A)\|$. Conclude: ϕ is an isomorphism of normed vector spaces. Thus we can speak of C^r-functions $Mat_{n\times n}(\mathbb{R}) \to \mathbb{R}$, for $0 \le k \le \infty$.

a. Verify: the following functions are polynomial.

i. $Mat_{n\times n}(\mathbb{R}) \stackrel{trace, det}{\to} \mathbb{R}$. ii. The coefficients $\{c_j(A)\}$ of the characteristic polynomial of A.

iii. The matrix product, $Mat_{n\times n}(\mathbb{R}) \times Mat_{n\times n}(\mathbb{R}) \to Mat_{n\times n}(\mathbb{R})$, $(A,B) \to A\cdot B$.

b. Verify: the inverse of a matrix, $GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$, $A \to A^{-1}$, is a C^ω-function.

c. Which of the following subsets of $Mat_{n\times n}(\mathbb{R})$ are open/compact? i. $GL(n)$ ii. $SL(n)$ iii. $O(n)$.

d. Let $\Sigma_{diag} \subset Mat_{n\times n}(\mathbb{R})$ be the subset of all the matrices that are diagonalizable over $\mathbb{C}$. (i.e. $U\cdot A\cdot U^{-1}$ is diagonal for some $U \in GL(n,\mathbb{C})$) Prove: any matrix whose eigenvalues are pairwise distinct complex numbers belongs to the interior $int(\Sigma_{diag})$.
(You can use the fact: if all the complex roots of a polynomial are distinct then locally they are C^ω-functions of the coefficients of the polynomial.)

e. Conclude: $\overline{int(\Sigma_{diag})} = Mat_{n\times n}(\mathbb{R})$. (Therefore $int(Mat_{n\times n}(\mathbb{R}) \setminus \Sigma_{diag}) = \varnothing$.)
(Because of this many engineers claim "Any matrix in real life is diagonalizable".)

f. Is $Mat_{n\times n}(\mathbb{R}) \setminus \Sigma_{diag}$ a closed subset of $Mat_{n\times n}(\mathbb{R})$? (Hint: look at $Mat_{2\times 2}(\mathbb{R})$)

2. Define the map $Mat_{n\times n}(\mathbb{R}) \stackrel{exp}{\to} Mat_{n\times n}(\mathbb{R})$ by $exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}$. (Convention: $A^0 = 1\!\!1$)

a. i. Compute $exp(A)$ for a diagonal matrix. (In particular verify that the series converges)

ii. Compute $exp(A)$ for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

b. Prove: the power series of $exp(A)$ converges absolutely, and the convergence is uniform on compact subsets of $Mat_{n\times n}(\mathbb{R})$.
You can use $||A\cdot B|| \le ||A||\cdot||B||$. (follows by Cauchy-Schwarz inequality)

c. Consider A as a complex matrix and take its Jordan form, $A = U^{-1}(Diag_A + Nilp_A)U$, where $U \in GL(n,\mathbb{C})$, $Diag_A$ is diagonal and $Nilp_A$ is strictly upper-triangular (corresponding to the Jordan cell structure). Verify: $Nilp^n_A = \mathbb{O}$ and $Diag_A \cdot Nilp_A = Nilp_A \cdot Diag_A$.
Prove: $exp(A) = U^{-1}\cdot exp(D_A)\cdot(\sum_{k=0}^{n} \frac{C^k_A}{k!})\cdot U$. (You will have to open the brackets/to change the order of summation in the series. Justify these steps.)

d. Prove: if A, B commute then $exp(A+B) = exp(A)exp(B)$.

e. Fix some $A \in Mat_{n\times n}(\mathbb{R})$ and define "a path" $\mathbb{R}^1 \stackrel{\gamma}{\to} Mat_{n\times n}(\mathbb{R})$, by $\gamma(t) = exp(t\cdot A)$. Compute $\frac{d\gamma}{dt}$.

3. a. We have defined the germs of sets and functions via equivalence relations. Verify: these are indeed equivalence relations.

b. Take the germ of a function $(\mathbb{R}^n, o) \stackrel{f_o}{\to} \mathbb{R}^1$. Write the full definition for the condition $f \ge 0$.

c. Define the basic operations on a finite number of germs:

$$\bigcap_i (X_i, o) := (\bigcap_i X_i, o), \qquad \bigcup_i (X, o) := (\bigcup_i X_i, o), \qquad (X,o)\setminus(Y,o) := (X\setminus Y, o), \qquad \prod_i (X_i, 0) := (\prod_i X_i, o).$$

Verify: the results are well defined, i.e. do not depend on the choice of representatives.

d. Let $0 \le r \le \infty$, and take $f \in C^r(\mathbb{R}^n, x_o)$. Verify: the derivatives of all orders, $f^{(j)}|_{x_o}$, $j = 0, \dots, r$, are well defined. (i.e. they do not depend on the choice of a representative of f)

e. Verify: the set $C^r(\mathbb{R}^p, o)$ is a commutative, unital ring.