Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 10.

Submission date: 10.06.2025. Questions to submit: 1.b.i. 3.b. 3.d. 3.e. 4.b. 5.

(Either typed or in readable handwriting and scanned in readable resolution.)

All vector spaces are finite-dimensional, over a field $\mathbb{k}$ of $char(\mathbb{k})=0$. The maps are linear. $dim_{\mathbb{k}}(V)=n$.

1. a. For $\sigma\in S_k$ we have defined $\sigma(v_1\otimes\cdots\otimes v_k)=v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}$ and claimed "This extends (by linearity) to the action $\sigma \circlearrowright \otimes^k V$." Write the details. Verify: this action preserves all the relations of $\otimes^k V$.
   b. Fix some bases: $V=Span\{v_i\}$, $W=Span\{w_i\}$.
      i. Prove: $\{v_i\otimes w_j\}$ is a basis of $V\otimes W$. The non-trivial part here is the linear independence. (Hint: use the dual bases of V^*, W^* and $V^*\otimes W\subseteq(V\otimes W)^*$.)
      ii. Write explicit bases for $Sym^k V^*$, $\overset{k}{\Lambda} V^*$. (Verify that these are bases.)
   c. Establish the natural (basis-free) isomorphisms: i. $V^*\otimes W\cong Hom(V,W)$. ii. $V_1\otimes(V_2\otimes V_3)\cong(V_1\otimes V_2)\otimes V_3$.
   iii. $V\otimes(U\oplus W)\cong(V\otimes U)\oplus(V\otimes W)$. iv. $V^*\otimes W^*\cong Hom(V,W^*)\cong Hom(V\otimes W,\mathbb{k})\cong(V\otimes W)^*$.

2. a. Prove: the symmetrization $\otimes^k V\overset{Sym}{\to}\otimes^k V$ and alternation $\otimes^k V\overset{Alt}{\to}\otimes^k V$ are projectors. Namely: $Sym\circ Sym=Sym$ and $Alt\circ Alt=Alt$.
   b. Fix k and compute $\lim\limits_{n\to\infty}\frac{dim[\otimes^k V]}{dim[Sym^k V]+dim[\wedge^k V]}$. Thus "About [...] of all k-linear forms are (skew)symmetric."
   c. Prove: i. $Sym(f\otimes g)=Sym(f\otimes Sym(g))=Sym(Sym(f)\otimes g)$.
   ii. $Alt(f\otimes g)=Alt(f\otimes Alt(g))=Alt(Alt(f)\otimes g)$.
   d. Given vectors $\{v_i\}$ in V prove: $v_1\wedge\cdots\wedge v_k=0$ iff these vectors are linearly dependent.
   e. To an element $l\in V^*$ associate the linear function $l(x)\in\mathbb{k}[\underline{x}]_1$ (homogeneous polynomials of degree=1). Accordingly we define the map $Sym^k(V^*)\ni Sym(l_1\otimes\cdots\otimes l_k)\to l_1(x)\cdots l_k(x)\in\mathbb{k}[\underline{x}]_k$. Prove: this extends to an isomorphism of commutative graded algebras, $\oplus_k Sym^k(V^*)\cong\mathbb{k}[\underline{x}]$.

3. a. Verify: $f\wedge f=0$ for any $f\in\overset{k}{\Lambda} V^*$ with k-odd.
   b. Given $f_i\in\overset{k_i}{\Lambda} V^*$, verify: $f_1\wedge\cdots\wedge f_k=\frac{(k_1+\cdots k_r)!}{k_1!\cdots k_r!}Alt(f_1\otimes\cdots\otimes f_r)$.
   c. Verify the basic properties of the exterior product: i. $(f\wedge g)\wedge h=f\wedge(g\wedge h)$.
   ii. $(f+g)\wedge h=f\wedge g+f\wedge h$. iii. $f\wedge g=(-1)^{k\cdot l}g\wedge f$, for $f\in\overset{k}{\Lambda} V^*$ and $g\in\overset{l}{\Lambda} V^*$.
   d. Take the exterior algebra $\Lambda V^*=\bigoplus_k\overset{k}{\Lambda} V^*$. Compute $dim_{\mathbb{k}}(\Lambda V^*)$.
   e. Given vectors $\{v_j\}$ in V and 1-forms $\{l_i\}$ in V^* prove: $(l_1\wedge\cdots\wedge l_k)(v_1\otimes\cdots\otimes v_k)=det\{l_i(v_j)\}_{i,j}$.

4. a. Prove: any endomorphism $\phi\circlearrowright V$ induces endomorphisms $\overset{k}{\Lambda}\phi\circlearrowright\overset{k}{\Lambda} V$ for $k=0,1,\dots$. (The maps $\overset{k}{\Lambda}\phi$ are natural, i.e. defined without fixing a basis.)
   b. In particular, the map $\overset{n}{\Lambda}\phi\circlearrowright\overset{n}{\Lambda} V$ is the multiplication by a scalar, denote it $f(\phi)$. Prove: $f(\phi)=det[\phi]$ where $[\phi]$ is the presentation matrix of ϕ in some/any basis of V.
   c. Deduce: $det[\phi]$ does not depend on the choice of basis. Deduce: $det[U\cdot A\cdot U^{-1}]=det[A]$ for any $A\in Mat_{n\times n}(\mathbb{k})$ and $U\in GL(n,\mathbb{k})$.
   d. Given two endomorphisms, $\phi_1,\phi_2\circlearrowright V$, study the scalar $f(\phi_1\circ\phi_2)$. Deduce: $det[A\cdot B]=det[A]\cdot det[B]$.
   e. Given a map $V\overset{\phi}{\to}W$ and the corresponding maps $\overset{k}{\Lambda} V\overset{\overset{k}{\Lambda}\phi}{\to}\overset{k}{\Lambda} W$, construct the dual maps $\overset{k}{\Lambda} W^*\overset{\overset{k}{\Lambda}\phi^*}{\to}\overset{k}{\Lambda} V^*$.
   f. Verify: $(l_1\wedge\cdots\wedge l_k)\big((\overset{k}{\Lambda}\phi)(v_1\wedge\cdots\wedge v_k)\big)=\Big((\overset{k}{\Lambda}\phi^*)(l_1\wedge\cdots\wedge l_k)\Big)(v_1\wedge\cdots\wedge v_k)$. Deduce: $det[A]=det[A^t]$.

5. Compute $\phi^*\omega$ for $\omega=zdx\wedge dy+ydz\wedge dx\in\Omega^2(\mathbb{k}^3)$ and $\mathbb{k}^2\ni(u,v)\overset{\phi}{\to}(cos(u),sin(u),v^7)\in\mathbb{k}^3$.