Geometric Calculus 2, 201.1.1041

Homework 11

Spring 2025 (D.Kerner)

Submission date: 17.06.2025. Questions to submit: 1.d. 2.b. 3.c. 3.f. 4.a. 4.b. 5.b.
(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Let $f(\underline{x}) = ln(1+\|x\|^2)$, $g(\underline{x}) = sin(\|x\|)$ for $\underline{x} \in \mathbb{R}^3$. Expand $df \wedge dg$ in the basis of $dx_i \wedge dx_j$.
   b. Express the restriction of $dy \wedge dz|_{S^2}$ in polar coordinates on S^2.
   c. Let $S^2 = S^2_+ \cup S^2_-$, and take the standard charts with the transition map $\mathbb{R}^2_{\underline{x}} \ni \underline{x} \to \underline{y} = \frac{\underline{x}}{\|\underline{x}\|^2} \in \mathbb{R}^2_{\underline{y}}$. Compute $\phi^*(dy_1 \wedge dy_2)$
   d. For $\mathbb{R}^n_x \supset \mathcal{U}_x \xrightarrow{\phi} \mathcal{U}_y \subset \mathbb{R}^n_y$ compute $\phi^*(dy_1 \wedge \cdots \wedge dy_n)$.
   e. Let $X \overset{\phi}{\hookrightarrow} Y$, $dimX = n$. Prove: $\phi^*\Omega^k(Y) = 0$ for $k > n$.
   f. The set of zeros of a differential form is defined as $Zeros(\omega) := \{p \in X |\ \omega_p = 0 \in \wedge^k T^*_{(X,p)}\}$. Give an equivalent definition of $Zeros(\omega)$ in coordinates. How $Zeros(\omega)$ are related to $Zeros(\phi^*\omega)$?
   g. Prove: any forms $\omega, \tilde{\omega} \in \Omega^n(X)$ are linearly dependent, i.e. $a\cdot\omega = \tilde{a}\cdot\tilde{\omega}$, where $Zeros(a) \subseteq Zeros(\tilde{\omega})$ and $Zeros(\tilde{a}) \subseteq Zeros(\omega)$.

2. Suppose a submanifold $X \subset \mathbb{R}^{n+r}$ is defined by equations $g_1(\underline{x}) = 0 = \cdots = g_r(\underline{x})$.
   a. (Dis)Prove: $dim(X) \geq n$.
   b. (Lagrange's theorem on conditional extremum) Suppose the forms $dg_1, \ldots, dg_r \in \Omega^1(\mathbb{R}^{n+r})$ are linearly independent at each point of X. Prove: if a function $f \in C^1(\mathbb{R}^{n+r})$, restricted to X, has a local extremum at $p \in X$, then $(df \wedge dg_1 \wedge \cdots \wedge dg_r)|_p = 0$. What is the relation to Calculus.3?

3. a. For $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z$ and the corresponding pullbacks of Ω^k prove: $\phi^* \circ \psi^* = (\phi \circ \psi)^*$.
   b. For a diffeomorphism $\phi \circlearrowright X$ prove: all the pullbacks $\phi^* \circlearrowright \Omega^k(X)$ are (linear) isomorphisms.
   c. For $\omega, \tilde{\omega} \in \Omega^1(Y)$ and a morphism $X \xrightarrow{\phi} Y$ prove $\phi^*(\omega \wedge \tilde{\omega}) = \phi^*(\omega) \wedge \phi^*(\tilde{\omega})$.
   d. Extend this to $\omega_k \in \Omega^k(X)$ and $\omega_l \in \Omega^l(X)$.
   e. Take an embedding $\mathbb{R}^2_{xy} \supseteq \mathcal{U} \overset{\phi}{\hookrightarrow} \mathbb{R}^3_{xyz}$. Verify: $\phi^*(dy \wedge dz) = 0$, $\phi^*(dx \wedge dz) = 0$. Do this in as many ways as possible, e.g. by checking the action on $T\mathcal{U}$, by computing the pullback in coordinates, by using $\phi^* d(\ldots) = d\phi^*(\ldots)$.
   f. Fix functions $f_1 \ldots f_k \in C^1(X)$ and vector fields $\xi_1 \ldots \xi_k$. Prove: $(df_1 \wedge \cdots \wedge df_k)(\xi_1, \ldots, \xi_k) = det[\{df_i(\xi_j)\}]$.
   g. Take a map $(\mathbb{R}^n_x, o) \xrightarrow{\phi} (\mathbb{R}^m_y, o)$, $x \to y(x)$. Let $\omega = \sum \omega_{i_1 \ldots i_k} dy_{i_1} \wedge \cdots \wedge dy_{i_k} \in \Omega^k(\mathbb{R}^m, o)$. Prove: $\phi^*\omega = \sum \phi^*(\omega_{i_1 \ldots i_k}) \cdot det[\frac{\partial(y_{i_1} \ldots y_{i_k})}{\partial(x_{j_1} \ldots x_{j_k})}] \cdot dx_{j_1} \wedge \cdots \wedge dx_{j_k}$.

4. (Below $1 \leq r \leq \infty$)
   a. For an embedded manifold $X \subset \mathbb{R}^N$ prove: $\Omega^k(\mathbb{R}^N) \twoheadrightarrow \Omega^k(X)$. (Recall the case $k=1$.)
   b. Prove: for any choice of charts $X = \cup \mathcal{U}_\alpha$, each $\omega \in \Omega^k(X)$ is presentable as $\sum_{\alpha,i} C_{\alpha,i} dg_{\alpha,i,1} \wedge \cdots \wedge dg_{\alpha,i,k}$, where $C_{\alpha,i}, g_{\alpha,i,j} \in C^r(X)$ and $supp(C_{\alpha,i}) \subseteq \mathcal{U}_\alpha$.

5. a. For $\Omega^\bullet(\mathcal{U}) \xrightarrow{d} \Omega^{\bullet+1}(\mathcal{U})$ prove: $d(\omega_k \wedge \omega_l) = \cdots$, $d(\phi^*\omega) = \phi^*(d\omega)$, and $d \circ d = 0$.
   b. For $\Omega^\bullet(X) \xrightarrow{d} \Omega^{\bullet+1}(X)$ prove: $d(\omega_k \wedge \omega_l) = \cdots$, $d(\phi^*\omega) = \phi^*(d\omega)$, and $d \circ d = 0$.
   c. Given $\omega \in \Omega^1(X)$ and ξ_1, ξ_2 vector fields on X prove: $d\omega(\xi_1, \xi_2) = \xi_1(\omega(\xi_2)) - \xi_2(\omega(\xi_1)) - \omega([\xi_1, \xi_2])$.