

Geometric Calculus 2, 201.1.1041

Homework 12

Spring 2025 (D.Kerner)

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Questions to submit: 1.a. 1.b. 2.d. 3.b. 3.c.ii. 3.c.iii. 5.a.

(Either typed or in readable handwriting and scanned in readable resolution.)

- Given $\omega \in \Omega^{k-1}(X)$ and vector fields $\xi_1 \dots \xi_k$ verify: $(d\omega)(\xi_1, \dots, \xi_k) = \sum (-1)^{j-1} \xi_j(\omega(\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_k))$.
 - Take a form with no zeros $\omega \in \Omega^{n+1}(\mathbb{R}^{n+1})$, a vector field ξ on \mathbb{R}^{n+1} , and the form $\omega(\xi)|_X \in \Omega^n(X)$. For $p \in X$ verify: $(\omega(\xi)|_X)_p = 0$ iff ξ is tangent to X at p .
 - For the radial vector field $\xi = \sum x_i \frac{\partial}{\partial x_i}$ deduce: the form $(dx_1 \wedge \dots \wedge dx_{n+1})(\xi)|_{S^n} \in \Omega^n(S^n)$ has no zeros. [This form is said to define the *outer* orientation]
- Two bases \hat{v}_\bullet and \hat{w}_\bullet of \mathbb{R}^n are called “of the same orientation” if the transition matrix $[\mathbb{I}]_{\hat{w}_\bullet}^{\hat{v}_\bullet}$ has $\det > 0$. Verify: the orientation is preserved under continuous change of basis. Which matrices of $GL(n) \circlearrowleft \mathbb{R}^n$ are orientation-preserving?
 - Take the reflection map $\mathbb{R}^N \ni x \xrightarrow{\phi} -x \in \mathbb{R}^N$. Suppose an oriented submanifold $X \subset \mathbb{R}^N$ (of $\dim(X) = n$) is invariant under ϕ , i.e. $\phi(X) = X$. Does ϕ preserve the orientation of X ?
 - Take $S^n \subset \mathbb{R}^{n+1}$ and the charts $S^n \setminus \{\hat{x}_{n+1}\}$, $S^n \setminus \{-\hat{x}_{n+1}\}$, with the standard coordinates. Compute the sign of the Jacobian for the chart transformation. [Deduce: S^n is orientable.]
 - Take a manifold with two charts $X = \mathcal{U}_1 \cup \mathcal{U}_2$, suppose $\mathcal{U}_1 \cap \mathcal{U}_2$ is path-connected. Prove: X is orientable.
 - Let X be orientable with d connected components. How many different orientations it has?
- Take a vector subspace $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ and a vector $\xi \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n$. Prove: two bases v_\bullet, w_\bullet of \mathbb{R}^n have the same orientation iff the bases $\{v_\bullet, \xi\}$ and $\{w_\bullet, \xi\}$ of \mathbb{R}^{n+1} have the same orientation.
 - Prove: a smooth hypersurface $X \subset \mathcal{U} \subseteq \mathbb{R}^{n+1}$ is orientable iff it is possible to choose unit normals $\{\hat{n}_p\}_{p \in X}$ that vary continuously with p .
 - Suppose the boundary of an open subset $\mathcal{U} \subset \mathbb{R}^{n+1}$ is a smooth hypersurface. (Is this always the case?) Moreover, suppose for each point $p \in \partial\mathcal{U}$ there exists a normal \hat{n}_p to $\partial\mathcal{U}$ pointing outside of \mathcal{U} . (Is this always the case?) Prove: in this case $\partial\mathcal{U}$ is orientable. (Choosing the outer normal one gets the outer orientation.)
 - (For $r \neq \omega$) Prove: there exists a vector field ξ on \mathbb{R}^{n+1} whose values on $\partial\mathcal{U}$ are exactly the outer normals. The corresponding orientation $(dx_1 \wedge \dots \wedge dx_{n+1})(\xi)$ is called the outer orientation.
 - Take a subset $X = \{f(x) = 0\} \subset \mathbb{R}^{n+1}$. Suppose f' has no zeros on X . Prove: X is an orientable hypersurface, and $X = \partial\mathcal{U}$, as above.
 - Take the parametrization $\mathbb{R}_{x_1 \dots x_n}^n \supset Ball_1(o) \xrightarrow{\sim} S_{x_{n+1} > 0}^n$. Which orientation is induced on $Ball_1(o)$ by the outer orientation on S^n ?
 - Take the ϕ, θ coordinates on $Torus \subset \mathbb{R}^3$. Which orientation is induced on $(0, 2\pi)_\phi \times (0, 2\pi)_\psi$ by the outer orientation of $Torus$?
- Prove: the Möbius strip is non-orientable.
 - Suppose $X \subset Y$, with $\dim(X) = \dim(Y)$, and X is non-orientable. Prove: Y is non-orientable. Deduce: the Klein bottle and the real projective plane ($\mathbb{R}P^2$) are non-orientable.
 - Deduce: the Möbius strip, Klein bottle, $\mathbb{R}P^2$ cannot be presented as $\{f(x) = 0\} \subset \mathcal{U} \subseteq \mathbb{R}^3$, where $f'|_X$ has no zeros.
- Given two manifolds $X \subset \mathbb{R}^{N_x}$, $Y \subset \mathbb{R}^{N_y}$, prove: $X \times Y \subset \mathbb{R}^{N_x + N_y}$ is a manifold.
 - Take the projections $X \xrightarrow{\pi_x} X \times Y \xrightarrow{\pi_y} Y$. To the forms $\omega_X \in \Omega^{k_x}(X)$, $\omega_Y \in \Omega^{k_y}(Y)$ associate $\pi_x^*(\omega_X) \wedge \pi_y^*(\omega_Y) \in \Omega^{k_x + k_y}(X \times Y)$. Express the zeros of $\pi_x^*(\omega_X) \wedge \pi_y^*(\omega_Y)$ via those of ω_X, ω_Y .
 - Prove: if X, Y are compact/path-connected/orientable, then so is $X \times Y$.