Geometric Calculus 2, 201.1.1041

Homework 13

Spring 2025 (D.Kerner)

Not for submission.

1. (For $1 \le r \le \infty$) Take a smooth orientable hypersurface $X \subset \mathbb{R}^{n+1}$. Prove: X can be (globally!) defined by $f(x) = 0$, where f has no critical points on X.

2. Let $\mathcal{U} \subseteq \mathbb{R}^n$. Define the maps: $C^{r-1}(\mathcal{U}) \stackrel{\psi_n}{\to} \Omega^n(\mathcal{U})$, by $f \to \omega^n_f := f \cdot dx_1 \wedge \cdots \wedge dx_n$,
   $\{\text{vector fields on } \mathcal{U}\} \stackrel{\psi_{n-1}}{\to} \Omega^{n-1}(\mathcal{U})$, by $\vec{F} \to \omega^{n-1}_{\vec{F}} := (dx_1 \wedge \cdots \wedge dx_n)(\vec{F})$,
   and $\{\text{vector fields on } \mathcal{U}\} \stackrel{\psi_1}{\to} \Omega^1(\mathcal{U})$, by $\vec{F} \to \omega^1_{\vec{F}} := \sum F_i \cdot dx_i$.
   a. Verify: these are $C^{r-1}(\mathcal{U})$-linear isomorphisms. Verify: $\omega^1_{grad(f)} = df$ and $\omega^{n-1}_{\vec{F}} \wedge \omega^1_{\vec{G}} = \omega^n_{\vec{F}\cdot\vec{G}}$.
   b. Specify bases (of the source and the target) for which $[\psi_1] = 1\!\!1$, and $[\psi_{n-1}] = 1\!\!1$.
   c. The classical "divergence operator" is defined via the identity $\omega^n_{div(\vec{F})} = d\omega^{n-1}_{\vec{F}}$.
   Verify: $div(\vec{F}) = \nabla \cdot \vec{F}$. (The scalar product.)
   d. For $\mathcal{U} \subseteq \mathbb{R}^3$ the vector product can be defined via $\omega^2_{\vec{F}\times\vec{G}} = \omega^1_{\vec{F}} \wedge \omega^1_{\vec{G}}$. Write $\vec{F} \times \vec{G}$ explicitly.
   The classical "rotor operator" is defined via $\omega^2_{rot(\vec{F})} = d\omega^1_{\vec{F}}$. Verify: $rot(\vec{F}) = \nabla \times \vec{F}$, the vector product.

3. a. Take a smooth hypersurface-germ $(X, p) \subset (\mathbb{R}^{n+1}, p)$. Fix some linearly independent vectors $v_1 \dots v_n \in T_{(X,p)}$. Define the 1-form $\omega_{v_1 \dots v_n} \in T^*_{(\mathbb{R}^{n+1},p)}$ by $\omega_{v_1\dots v_n}(v) = det[v_1, \dots, v_n, v]$. For the map ψ_1 of question 2 prove: $(\psi_1)^{-1}\omega_{v_1\dots v_n}$ is the normal $\vec{n}_p$ to X at p.
   b. Given a function $\mathbb{R}^n \supset \mathcal{U} \stackrel{f}{\to} \mathbb{R}$, take its graph $\Gamma_f \subset \mathcal{U} \times \mathbb{R}^1_y$. Take the upper normal to Γ_f, i.e. $\hat{n} \cdot \hat{y} > 0$, and the corresponding orientation of Γ_f.
   i. What is the corresponding orientation on $\mathcal{U}$?
   ii. Given a vector field $\vec{F}$ on $\mathcal{U} \times \mathbb{R}^1_y$, prove: its flux through Γ_f is $\int_{\mathcal{U}} (\vec{F} \cdot (-\nabla_u f, 1))|_{\Gamma_f} \cdot du_1 \cdots du_n$.
   c. Take $S^2 \subset \mathbb{R}^3$, with the outer normal. (What is the corresponding order of θ, ϕ?)
   i. Prove: the flux of the field $\vec{F} = f \cdot \vec{r}$ through S^2 equals to $\iint_S \vec{F} d\vec{S} = \iint_{\theta,\phi} f \cdot r^3 sin(\theta) d\theta d\phi$.
   ii. Compute the flux of $\vec{F} = \frac{\vec{r}}{r^d}$ through $S^2 \subset \mathbb{R}^3$.

4. a. Take a smooth hypersurface $X \subset \mathbb{R}^{n+1}$ with a chosen normal $\hat{n}$. Project onto coordinate hyperplanes, $\mathbb{R}^{n+1} \subset X \stackrel{\pi_i}{\to} \mathbb{R}^n = \{x_i = 0\}$. Denote the images $\mathcal{U}_i := \pi_i(X)$. Suppose each π_i is invertible and (outside of a set of $dim \le n-1$) the inverse map $\pi_i^{-1} : \mathcal{U}_i \to X$ is a parametrization.
   Prove: the flux of a vector field $\vec{F}$ through X equals to $\sum_i \int_{\mathcal{U}_i} sign(n_i) \cdot (F_i \circ \pi_i^{-1}) \cdot du^i_1 \cdots du^i_n$.
   b. Let $X = \partial\overline{\mathcal{U}} \subset \mathbb{R}^{n+1}$, where $\overline{\mathcal{U}}$ is a convex bounded set. Suppose each F_i is independent of x_i.
   Prove: the total flux of $\vec{F}$ through X is zero.
   c. Take $S = (\partial Pyramid) \setminus \mathcal{U}$, with $Pyramid \subset \mathbb{R}^3$ defined by $x, y, z \ge 0$, $x + y + z \le 1$, and $\mathcal{U} \subset \mathbb{R}^2_{xy}$ is defined by $x, y \ge 0$, $x^2 + y^2 \le \frac{1}{\sqrt{2}}$. (The orientation corresponds to the outer normal.)
   Compute $\int_S \omega$, where $\omega = (x + z) dx \wedge dy + (z + y + cos(x)) dy \wedge dz + (x - sin(y)) dz \wedge dx|_S$.

5. a. Compute $\int_C \vec{F} d\vec{C}$ in the following cases: i. $\vec{F} = \frac{(-y,x)}{x^2+y^2}$, $C = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\} \subset \mathbb{R}^2$ (counterclockwise).
   ii. $\vec{F} = \frac{(-y,x)}{x^2+y^2}$, and the curve $(\sqrt{3}, 1) \rightsquigarrow (-\sqrt{3}, 1)$ is given in polar coordinates by $r(\theta) = \frac{1}{1 - sin(\theta)}$.
   b. For each $n \in \mathbb{N}$ give a closed oriented curve that does not pass through $(0,0)$ and satisfies: $\oint_{\vec{C}} \frac{(-y,x)}{x^2+y^2} d\vec{C} = 2\pi n$.
   c. Find the smooth, closed oriented curve C for which the integral $\oint_C (\frac{x^2 y}{4} + \frac{y^3}{3}) dx + x dy$ achieves maximum.
   d. Take a compact $\bar{\mathcal{U}} \subset \mathbb{R}^2$, suppose $\partial\bar{\mathcal{U}}$ is piecewise smooth. Take the outer orientation. Prove: $vol_2(\bar{\mathcal{U}}) = \oint_{\partial\bar{\mathcal{U}}} x dy = -\oint_{\partial\bar{\mathcal{U}}} y dx$. Compute the area bounded by the curve $|\frac{x}{a}|^{\frac{2}{p}} + |\frac{y}{b}|^{\frac{2}{p}} = 1$.