Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 2.

Submission date: 7.04.2025. Questions to submit: 1.a. 1.d. 2.c. 4. 5.b. 5.c. 6.c.
(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Suppose $(\mathbb{R}^n, o) \overset{f}{\to} \mathbb{R}^m$ is C^1 and $rank[f'|_o] = k$. Prove: $rank[f'|_x] \geq k$ for x close to o.
(Namely: for any representative $\mathcal{U} \overset{f}{\to} \mathbb{R}^m$ there exists a neighborhood $o \in \tilde{\mathcal{U}} \subseteq \mathcal{U}$ such that ...)

b. Prove the lemma on the normal forms of immersions and submersions, see question 7 of homework 0. (We have proved one of these in the class.)

c. Deduce the open mapping theorem: if $\mathbb{R}^n \supseteq \mathcal{U} \overset{f}{\to} \mathbb{R}^m$ is C^1, $m \leq n$, and $rank[f'] = m$ everywhere on $\mathcal{U}$ then f sends open sets to open sets.

d. Let $\mathbb{R}^n \supseteq \mathcal{U} \overset{\phi}{\to} \tilde{\mathcal{U}} \subseteq \mathbb{R}^{\tilde{n}}$ be a C^1-diffeomorphism of open subsets. Prove: $n = \tilde{n}$.

2. Prove: the following sets are C^ω-manifolds. Try to imagine/to visualise these manifolds, e.g. $X_2 \cong S^1 \times S^1$.

a. $X_n = \{(x, y)|\ \|x\| = \|y\| = 1\} \subset \mathbb{R}^n \times \mathbb{R}^n$.

b. $\{(x, y, z)|\ x^2 + y^2 = 1,\ x^2 + y^2 + z^2 = 2x\} \subset \mathbb{R}^3$.

c. $\{(x_1, \ldots, x_4)|\ \|x\| = 1,\ x_1x_2 + x_3x_4 = 0\} \subset \mathbb{R}^4$.

3. Find the non-smooth points of the following sets:

a. $\{(x, y, z)|\ (x^2 + y^2 + z^2)^{17} = \frac{z^3}{(x^2+y^2)^8}\}$ b. $\{\underline{x}|\ ||L(\underline{x})|| = 1\} \subset \mathbb{R}^n$ for $L \in Hom(\mathbb{R}^n, \mathbb{R}^m)$.

c. $\{(x, y)|\ y^2 = \prod_{i=1}^{d}(x - c_i)^2\}$. Draw this curve in small neighborhoods of non-smooth points.

4. Identify $Mat_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, as in homework 1. Prove: the subsets $SL(n, \mathbb{R}) \subset GL(n, \mathbb{R}) \subset Mat_{n\times n}(\mathbb{R})$ are C^ω-manifolds. Determine their dimensions. Are they compact? Path-connected?

5. a. Prove: $(C, o) = \{(x, y)|\ x \cdot y = 0\} \subset (\mathbb{R}^2, o)$ is not the germ of a C^0-submanifold. [Done in the class.]

b. Prove: $C = \{(x, y)|\ y = |x|\} \subset \mathbb{R}^2$ is a C^0-submanifold, but not a C^1-submanifold.
Verify: $\mathbb{R}^1 \ni t \to (t^{101}, |t|^{101}) \in \mathbb{R}^2$ is a C^{100}-parametrization of C. Any contradiction?

c. Define the curve $C \subset \mathbb{R}^2$ by the parametrization $\mathbb{R}^1 \ni t \to (t^3, t^5) \in \mathbb{R}^2$. Prove: C is a C^1 manifold, but not a C^2-manifold. Give an (explicit) non-degenerate C^1-parametrization of C.

d. For each $r \geq 1$ give an example of C^r-manifold that is not a C^{r+1}-manifold.

e. Prove: the dimension of a path-connected manifold is well defined,i.e. $dim(X, x_o) = const$ for $x_o \in X$.

6. a. We gave four equivalent characterizations of smoothness of (X, o), and have proved: $1 \Rrightarrow 2, 3, 4$. Prove the other implications.

b. Define $f(x, y) = y^2$ and $X := \{f(x, y) = 0\} \subset \mathbb{R}^2$. Note: $f'|_{y=0} = 0$, at each point of X.
Is X a submanifold of $\mathbb{R}^2$? (Any contradiction to one of the characterizations of part a?)

c. Take a C^r-function, $\mathbb{R}^n \supseteq \mathcal{U} \overset{f}{\to} \mathbb{R}$. Prove: the graph $\Gamma_f \subset \mathcal{U} \times \mathbb{R}^1$ is a C^r-manifold, and it is *globally* rectifiable.

d. Prove: every manifold is locally path-connected at each point.