Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 3.

Submission date: 14.04.2025.

Questions to submit: 1.b.ii. 1.c.i. 1.c.ii. 2.c. 3.b. 3.c. 3.e.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. In the class we have stated several equivalent definitions for a function-germ $(X, x_o) \stackrel{f}{\to} \mathbb{R}$ to be C^r. Verify that they are all equivalent.

b. Given a C^2-function (on a manifold) $X \stackrel{f}{\to} \mathbb{R}^1$, define the notions of local min/max, critical point.

i. Verify: the notion of critical point is well defined. (It does not depend on the choice of parametrization, rectification.)

ii. Take a parametrization $(\mathbb{R}^n, o) \xrightarrow[\sim]{\phi} (X, x_o)$. Is the condition "$(f \circ \phi)^{(2)}|_o$ is positive definite" well defined? Is the condition "$(f \circ \phi)'|_o = 0$ and $(f \circ \phi)^{(2)}|_o$ is positive definite" well defined?

iii. State the criterion for the local min/max of f in terms of $(f \circ \phi)'|_o$ and $(f \circ \phi)^{(2)}|_o$.

c. Construct C^ω-diffeomorphisms:

i. $\mathbb{R}^n \xrightarrow{\sim} Ball_1(o) \subset \mathbb{R}^n$.

ii. $\mathbb{R}^n \setminus \{o\} \xrightarrow{\sim} \{x | \ 1 < \|x\| < 2\} \subset \mathbb{R}^n$.

iii. $Mat_{2\times 2}(\mathbb{R}) \supset SO(2) \cong S^1$,

iv. $Mat_{2\times 2}(\mathbb{R}) \supset O(2) \cong S^1 \amalg S^1$.

v. $Mat_{2\times 2}(\mathbb{R}) \supset SL(2, \mathbb{R}) \cong \{(x, y, z, w) | \ x^2 + y^2 = 1 + z^2 + w^2\} \subset \mathbb{R}^4$.

2. a. Take a subgerm $(X, x_o) \subset (\mathbb{R}^N, x_o)$ of C^1-manifold, and a subgerm $(\mathcal{U}, x_o) \subset (X, x_o)$. Prove: the following conditions are equivalent. In this case $(\mathcal{U}, x_o) \subset (X, x_o)$ is called an open subgerm.

i. For some/any rectification $(X, x_o) \stackrel{\Phi}{\to} (\mathbb{R}^n, o)$ the image $\Phi(\mathcal{U}, x_o) \subseteq (\mathbb{R}^n, o)$ is open.

ii. For some/any parametrization $(\mathbb{R}^n, o) \xrightarrow[\sim]{\phi} (X, x_o)$ the preimage $\phi^{-1}(\mathcal{U}, x_o) \subset (\mathbb{R}^n, o)$ is open.

iii. $(\mathcal{U}, x_o) = (\tilde{\mathcal{U}}, x_o) \cap (X, x_o)$ for an open subgerm $(\tilde{\mathcal{U}}, x_o) \subset (\mathbb{R}^N, x_o)$.

b. Let $X \subset \mathbb{R}^N$ a C^r-submanifold. A subset $\mathcal{U} \subset X$ is called open if all its germs are open. A subset $Z \subset X$ is called closed if $X \setminus Z$ is open.

i. Verify: every open subset of X is a C^r-submanifold.

ii. (Dis)Prove: $Z \subset X$ is closed iff $Z \subset \mathbb{R}^N$ is closed.

iii. (Dis)Prove: $Z \subset X$ is compact iff $Z \subset \mathbb{R}^N$ is compact.

c. Prove: S^n cannot be C^1-embedded into $\mathbb{R}^n$. [Hint: S^n is compact.]

3. Take a C^r-map from a submanifold $\mathbb{R}^N \supset X \stackrel{\phi}{\to} \mathbb{R}^M$. Suppose $rank[\phi'|_p] = dim X$ for each $p \in X$.

a. Verify: ϕ is a C^r-diffeomorphism locally at each point, i.e. $(X, p) \xrightarrow[\sim]{\phi|} \phi(X, p)$.

b. Does ϕ have to be (globally) injective?

c. Suppose ϕ is injective. Is ϕ an embedding? (I.e. $\phi(X) \subset \mathbb{R}^M$ is a submanifold and $X \xrightarrow[\sim]{\phi} \phi(X)$)

Hint: define $\mathbb{R}^1 \stackrel{\phi}{\to} \mathbb{R}^2$ by $\phi(t) = ((t - t^3) \cdot e^{-t^2}, t \cdot e^{-t^2})$. Verify that ϕ satisfies the assumptions. Draw the image. (You can use any software.)

d. Construct another example: an injective analytic map $\mathbb{R}^1 \to S^1 \times S^1$, with non-degenerate derivative, whose image is dense in $S^1 \times S^1$.

(You can use the fact: the sequence $\{n \cdot c \ mod \ \mathbb{Z}\}$ is dense in $[0, 1]$ provided $c \notin \mathbb{Q}$.)

e. Prove: if X is compact, $\phi \in C^r$ is injective, and $rank[\phi'|_p] = dim X$ everywhere, then $X \xrightarrow[\sim]{\phi \in C^r} \phi(X)$.

f. A "trefoil knot" in $\mathbb{R}^3$ can be defined via parametrization $x(t) = sin(t) + 2sin(2t)$, $y(t) = cos(t) - 2cos(2t)$, $z(t) = -3sin(3t)$. Draw this curve. [You can use any software.] Prove: $S^1 \xrightarrow[\sim]{C^\omega} Trefoil$.