

# Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

## Homework 4.

Submission date: 21.04.2025.

Questions to submit: 1.c.ii. 2.a. 2.b.ii. 3.b. 5.c. 6.c. 6.d.

(Either typed or in readable handwriting and scanned in readable resolution.)



1. a. Given  $C^r$ -submanifolds  $X \subset \mathbb{R}^n$ ,  $Y \subset \mathbb{R}^M$ , prove:  $X \times Y \subset \mathbb{R}^N \times \mathbb{R}^M$  is a  $C^r$ -submanifold.  
b. Take a function  $X \xrightarrow{f \in C^r} \mathbb{R}^m$  and its graph  $\Gamma_f \subset X \times \mathbb{R}^m$ . Prove:  $\Gamma_f \xrightarrow{C^r} X$ .  
c. Take the torus  $X = \{(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\} \subset \mathbb{R}^3$ . Restrict a function  $\tilde{f} \in C^1(\mathbb{R})$  to the torus,  $f := \tilde{f}|_X$ .  
i. Prove:  $f$  bounded and has at least two critical points. [See q.1. of hwk.3.]  
ii. For the case  $\tilde{f}(x, y, z) = x$  find and classify the critical points of  $f$ .  
d. For a map  $X \xrightarrow{\phi \in C^r} Y$  we have seen the pullback  $C^r(X) \xleftarrow{\phi^*} C^r(Y)$ .  
i. Verify:  $(Id_X)^* = Id_{C^r(X)}$  and  $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ .  
ii. Show by examples that functions cannot be  $C^r$ -pushed-forward, even if  $\phi$  is a bijection (but not a diffeomorphism).
2. a. Take a smooth hypersurface  $X = \{f(x) = 0\} \subset \mathbb{R}^{n+1}$ , here  $f'$  has no zeros on  $X$ . Denote by  $X_{||} \subset X$  the subset of points whose tangent hyperplanes are parallel to the axis  $\hat{x}_{n+1}$ . Write the defining equations of  $X_{||} \subset \mathbb{R}^{n+1}$ .  
b. We have defined  $T_{(X, x_o)}$  (geometrically) as the set of limiting lines, without using the smoothness of  $(X, x_o)$ . Compute  $T_{(X, x_o)}$  in the following (non-smooth) cases: i.  $X = \{y = |x|\} \subset \mathbb{R}^2$  ii.  $X = \{z^2 = x^2 + y^2\} \subset \mathbb{R}^3$ .  
c. We gave three presentations of  $T_{(X, x_o)}$ , when  $(X, x_o)$  is defined via equations, parametrizations and as the graph of a function. Verify: these presentations are equivalent.  
d. Take a submanifold  $(X, x_o) \subset (\mathbb{R}^N, x_o)$  and a vector  $v \in T_{(X, x_o)}$ .  
Prove: there exists a map  $(\mathbb{R}^1, o) \xrightarrow{\phi} (X, x_o)$  satisfying:  $\phi'|_o = v$ .
3. Fix a hypersurface-germ  $(X, x_o) = \{g(x) = 0\} \subset (\mathbb{R}^{n+1}, x_o)$ , with  $g'|_{x_o} \neq 0$ . Take a function  $(X, x_o) \xrightarrow{f} \mathbb{R}$ .  
a. Find the direction of the steepest ascent of  $f$ . Find the maximal rate of change of  $f$  (in this direction).  
b.  $f$  has an extremum at  $x_o$  if and only if its maximal rate of change vanishes. Write the corresponding equation.  
c. Consider this setup as “Extremum of  $f$  with the constraint  $g$ ”. Write the relevant equation from Calculus 3. Verify that it is the same as in b.
4. a. Given  $(X, x_o) \xrightarrow{f} \mathbb{R}$  and  $v \in T_{(X, x_o)}$  we gave three equivalent definitions of the directional derivative  $\partial_v f$ . (Via parametrizations of  $X$ , extensions of  $f$ , and “a geometric one”.) Verify: these are indeed equivalent.  
b. Given two submanifold-germs  $(X, x_o), (X', x_o) \subset (\mathbb{R}^N, x_o)$ , suppose  $(X, x_o) \cap (X', x_o)$  is also a manifold. (Dis)prove:  $T_{(X, x_o) \cap (X', x_o)} = T_{(X, x_o)} \cap T_{(X', x_o)}$ .
5. a. Define the function  $Mat_{n \times n}(\mathbb{R}) \supset Ball_1(\mathbb{O}) \xrightarrow{\ln} Mat_{n \times n}(\mathbb{R})$  by  $\ln(\mathbb{I} - A) := -\sum_j \frac{A^j}{j}$ . Prove: this power series converges absolutely on  $Ball_1(\mathbb{O})$ . Deduce:  $\ln$  is  $C^\omega$  on  $Ball_1(\mathbb{O})$ . (See hwk.1.)  
b. Suppose  $A = U \cdot A_{diag} \cdot U^{-1}$ , and  $A, A_{diag} \in Ball_1(\mathbb{O})$ . Prove:  $\ln(\mathbb{I} + A) = U \cdot \ln(\mathbb{I} + A_{diag}) \cdot U^{-1}$ .  
c. Prove:  $\exp[\ln(\mathbb{I} + A)] = \mathbb{I} + A$ . Assuming  $\exp(A) \in Ball_1(\mathbb{I})$ , prove:  $\ln[\exp(A)] = A$ .  
d. Prove:  $\det[\exp(A)] = \exp[\text{trace}(A)]$ .  
e. Compute the total derivative (differential) for the following maps:  
i. The  $k$ -th power map  $Mat_{n \times n}(\mathbb{R}) \ni A \xrightarrow{\phi} A^k \in Mat_{n \times n}(\mathbb{R})$ . ii. The inverse map  $GL(n, \mathbb{R}) \ni A \xrightarrow{\phi} A^{-1} \in GL(n, \mathbb{R})$ . (Express the answer as a linear form, do not try to write down the partial derivatives.)
6. A group  $G$  is called “a Lie group” if it is a  $C^r$ -manifold and the group operations (the product and the inverse) are  $C^r$ -maps.  
a. Verify:  $GL(n)$  is a  $C^\omega$ -Lie group.

- b. Given a subgroup  $G < GL(n)$ , take its germ at the unit matrix  $(G, \mathbf{I})$ . Prove: if  $(G, \mathbf{I})$  is a manifold, then the whole group  $G$  is a manifold. [Hint: to check the smoothness of  $(G, A_o)$  apply the map  $Mat_{n \times n}(\mathbb{R}) \xrightarrow{\cdot A_o^{-1}} Mat_{n \times n}(\mathbb{R})$ .]
- c. Prove: the groups  $SL(n, \mathbb{R})$ ,  $SO(n, \mathbb{R})$  are  $C^\omega$ -Lie groups.
- d. Prove the following identifications of the tangent spaces at the unit matrix,  $T_{\mathbf{1}}G \subseteq Mat_{n \times n}(\mathbb{R})$ :
- $\mathfrak{gl}(n, \mathbb{R}) := T_{\mathbf{1}}GL(n, \mathbb{R}) = Mat_{n \times n}(\mathbb{R})$
  - $\mathfrak{sl}(n, \mathbb{R}) := T_{\mathbf{1}}SL(n, \mathbb{R}) = (\text{matrices with zero trace})$
  - $\mathfrak{so}(n, \mathbb{R}) := T_{\mathbf{1}}SO(n, \mathbb{R}) = (\text{skew-symmetric matrices})$
- e. A vector subspace  $V \subseteq Mat_{n \times n}(\mathbb{R})$  is called “a Lie sub-algebra” if it is closed under commutation, i.e.  $[A, B] := AB - BA \in V$  for all  $A, B \in V$ . Verify:  $\mathfrak{gl}(n, \mathbb{R})$ ,  $\mathfrak{sl}(n, \mathbb{R})$ ,  $\mathfrak{so}(n, \mathbb{R})$  are Lie sub-algebras of  $Mat_{n \times n}(\mathbb{R})$ .