Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 5.

Submission date: 2.05.2025. Questions to submit: 1.d.i. 1.d.ii. 2.c. 2.f. 3.a. 4.b.
(Either typed or in readable handwriting and scanned in readable resolution.)

Below X is a C^r-manifold, $1 \leq r \leq \infty, \omega$, $dim(X) = n$, with coordinate charts $X = \cup(\mathcal{U}_\alpha, \phi_\alpha)$.

1. a. Given two charts $X \supset \mathcal{U}_\alpha, \mathcal{U}_\beta$, with their coordinate maps $\mathcal{U}_\bullet \stackrel{\Phi_\bullet}{\rightsquigarrow} \Phi_\bullet(\mathcal{U}_\bullet) \subset \mathbb{R}^n$. Suppose $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing$. Construct (the natural) isomorphism of tangent bundles: $T\Phi_\alpha(\mathcal{U}_{\alpha\beta}) \rightsquigarrow T\Phi_\beta(\mathcal{U}_{\alpha\beta})$.
   b. Given some charts $X = \cup\mathcal{U}_\alpha$, with coordinate maps $\mathcal{U}_\alpha \stackrel{\Phi_\alpha}{\to} \tilde{\mathcal{U}}_\alpha \subset \mathbb{R}^n_\alpha$. Take functions $\tilde{\mathcal{U}}_\alpha \stackrel{f_\alpha \in C^r}{\to} \mathbb{R}$. What is the necessary and sufficient condition for this collection $\{f_\alpha\}$ "to combine" into an element of $C^r(X)$?
   c. For a submanifold $(X, x_o) \subset (\mathbb{R}^N, x_o)$ and a parameterized curve-germ $(\mathbb{R}^1, o) \ni t \to x(t) \in (X, x_o)$ take the velocity $\frac{dx(t)}{dt}|_{t=o} \in \mathbb{R}^N$. Prove: $T_{(X,x_o)}$ is the union of such velocity vectors over all the curve germs.
   d. Define the torus $X \subset \mathbb{R}^3$ by $(\sqrt{x^2+y^2} - R)^2 + z^2 = r^2$. Let $X \stackrel{f}{\to} S^2_{R+r}$ be the vertical projection into the lower hemisphere, i.e. $(x, y, z) \to (x, y, -\sqrt{(R+r)^2 - x^2 - y^2})$.
      i. Take a vector $v \in T_{X,p} \subset T_{(\mathbb{R}^3,p)}$. Write its image in $T_{(S^2,f(p))}$.
      ii. At which points of X is df of full rank? Where is df of $rank = 1$?
      iii. Is the image $f(X) \subset S^2_{R+r}$ a submanifold?

2. a. Let $S^{n-1} = \{\|x\| = 1\} \subset \mathbb{R}^n$. Write down the defining equation(s) of $TS^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n_v$.
   b. Take the projection $\mathbb{R}^n \times \mathbb{R}^n_v \supset TS^{n-1} \stackrel{\pi}{\to} \mathbb{R}^n_v$. Prove: π is surjective.
   c. Write down the defining equations of $TSO(n) \subset Mat_{n\times n}(\mathbb{R}) \times Mat_{n\times n}(\mathbb{R})$.
   d. For an open subset $\mathcal{U} \subset X$ define $TX|_\mathcal{U} := \amalg_{x\in\mathcal{U}}\{x\} \times T_{(X,x)}$ Verify: $TX|_\mathcal{U} = T\mathcal{U}$.
   e. Given $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$, verify: $T(X \times Y) = TX \times TY$.
   f. For the torus $X \subset \mathbb{R}^3$ prove: $TX \cong X \times \mathbb{R}^2$. Write the defining equations of $TX \subset \mathbb{R}^3 \times \mathbb{R}^3_v$.
   g. (Dis)Prove: an injection $X \hookrightarrow Y$ induces an injection $TX \hookrightarrow TY$.
   h. (Dis)Prove: a surjection $X \twoheadrightarrow Y$ induces a surjection $TX \twoheadrightarrow TY$.

3. a. Take the C^ω-vector fields, $\xi := sin(x)\frac{d}{dx}$, $\eta := x\frac{d}{dx}$, $\delta := x^2\frac{d}{dx}$ on $(a, b) \subset \mathbb{R}^1$.
   Consider the following cases: $(a, b) = (-\epsilon, \epsilon)$, $(a, b) = (-2\pi, 2\pi)$.
   For each of these cases, which of these vector fields are related by coordinate changes?
   b. Express the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ on $\mathbb{R}^2$ in the polar coordinates. Express the vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}$ in the cartesian coordinates.
   c. In the class we saw the "stereographic" covering $S^1 = \mathbb{R}^1_+ \cup \mathbb{R}^1_-$, with coordinates x_+, x_-. Write the pushforward (transition) map for vector fields from $\mathbb{R}^1_+$ to $\mathbb{R}^1_-$. Prove: if a vector field $a(x_+)\frac{d}{dx_+}$ is polynomial in both charts then the degree of this polynomial is ≤ 2.
   d. Take the covering $\mathbb{R} \to S^1$ by $t \to e^{it}$. Prove: any C^r-vector field on S^1 is presentable as $c(t)\frac{\partial}{\partial t}$ for some periodic function $c \in C^r(\mathbb{R}^1)$.

4. Take the polar coordinates (r, θ, ϕ) on $\mathbb{R}^3$, with $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$.
   a. Take the charts $S^2 = (S^2 \setminus \{-\hat{z}\}) \cup (S^2 \setminus \{\hat{z}\})$. Define the coordinate maps:
      - $S^2 \setminus \{-\hat{z}\} \to \mathbb{R}^2_+ := \{z = 1\} \subset \mathbb{R}^3$, by projection from the point $\{-\hat{z}\}$.
      - $S^2 \setminus \{\hat{z}\} \to \mathbb{R}^2_- := \{z = -1\} \subset \mathbb{R}^3$, by projection from the point $\{-\hat{z}\}$.

      Take the vector fields $\partial_{x_1}, \partial_{x_2}$ on $\mathbb{R}^2_+$. Write their pushforwards on $\mathbb{R}^2_-$. Write these vector fields in the polar coordinates on S^2.
   b. Do the vector fields $sin(\theta)\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$ extend to the global C^1-fields on S^2?