Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 6.

Submission date: 9.05.2025. Questions to submit: 1.c. 1.e. 2.c. 3.b. 4.a. 4.b. 5.a.ii.
(Either typed or in readable handwriting and scanned in readable resolution.)

1. Given a C^{r-1}-vector field ξ and $f \in C^r(X)$, we have defined the function $X \stackrel{\xi(f)}{\to} \mathbb{R}^1$ pointwise, $\xi(f)|_p := \partial_\xi f|_p$.
   a. Suppose in some chart ξ is presented as $(c_1(x), \dots, c_n(x))$. Verify (in that chart): $\xi(f) = \sum c_i(x)\partial_{x_i} f(x)$.
   b. Prove: ξ defines an $\mathbb{R}$-linear map $C^r(X) \stackrel{\xi}{\to} C^{r-1}(X)$. Is this a homomorphism of rings? Establish the Leibniz rule, $\xi(f \cdot g) = \cdots$
   c. Suppose $r \geq 2$. Define the commutator of two vector fields, $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$, via its action on functions, as in a. Prove $[\xi, \eta]$ is a C^{r-2}-vector field on X. Namely, this is a differential operator of order one, independent of coordinate choices.
   d. For vector fields ξ, η on X verify: $[f \cdot \xi, g \cdot \eta] = f \cdot g \cdot [\xi, \eta] - g \cdot \eta(f)\xi + f \cdot \xi(g)\eta$ for any $f, g \in C^1(X)$.
   e. Take the vector fields $\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}$ in polar coordinates in $\mathbb{R}^3$. Compute $[\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}]$. Express these vector fields in x, y, z-coordinates, and compute the corresponding commutator. (Do you get the same answer?)

2. Take a C^r map : $X \stackrel{\psi}{\to} Y$. (Dis)Prove:
   a. If ψ is injective then ψ_* is injective.
   b. If ψ is surjective then ψ_* is surjective.
   c. $\psi^*((\psi_*\xi)(f)) = \xi(\psi^*(f))$ for each $f \in C^r(Y)$.

3. A vector field ξ on Y is called tangent to a submanifold $X \subset Y$ if $\xi|_p \in T_{(X,p)}$ for each point $p \in X$.
   a. Verify: the restriction $\xi|_X$ is a vector field on X iff ξ is tangent to X.
   b. Take a subset $X = \{\underline{f}(x) = 0\} \subset \mathbb{R}^N$. Prove: ξ is tangent to X (at its smooth points) iff $\xi(\underline{f})|_X = 0$.
   c. Prove: if ξ, η are tangent to X then $\xi + \eta$ and $[\xi, \eta]$ are tangent as well.

4. a. Prove: TX is trivial iff there exist vector fields $\xi_1, \dots, \xi_n$ on X whose values $\xi_1|_{x_o}, \dots, \xi_n|_{x_o}$ are linearly independent for each $x_o \in X$. (We did one direction in the class.)
   b. Give a C^ω-vector field without zeros on S^1. Deduce (once again): TS^1 is trivial.
   c. Prove that TS^3 is trivial, as follows.
      i. Take the vector field $\xi_1 = -x_2\partial_1 + x_1\partial_2 - x_4\partial_3 + x_3\partial_4$ on $\mathbb{R}^4$. Verify: ξ_1 is tangent to $S^3 \subset \mathbb{R}^4$. Verify: the restriction $\xi_1|_{S^3}$ is a vector field on S^3 without zeros.
      ii. In a similar way write vector fields ξ_2, ξ_3 on $\mathbb{R}^4$ that restrict to vector fields on S^3, and such that $\xi_1|_{x_o}, \xi_2|_{x_o}, \xi_3|_{x_o}$ are linearly independent for each point $x_o \in S^3$. (Here one has to verify that certain 3×4 matrix of linear forms is non-degenerate everywhere on S^3. You can skip this.)

5. a. Take a coordinate chart $\mathbb{R}^N \supset X \supseteq \mathcal{U} \stackrel{\phi}{\rightsquigarrow} \tilde{\mathcal{U}} \subseteq \mathbb{R}^n$. Fix a sequence of points $\{x_\bullet\} \subset \mathcal{U}$. (Dis)Prove:
      i. The sequence $\{x_\bullet\}$ converges in $\mathcal{U}$ iff $\{\phi(x_\bullet)\}$ converges in $\tilde{\mathcal{U}}$.
      ii. $\{x_\bullet\}$ is a Cauchy sequence of points in $\mathcal{U}$ iff $\{\phi(x_\bullet)\}$ is a Cauchy sequence of points in $\tilde{\mathcal{U}}$.
   b. Take some coordinate charts, $X = \cup\mathcal{U}_\alpha$, with $\mathcal{U}_\alpha \stackrel{\phi_\alpha}{\rightsquigarrow} \tilde{\mathcal{U}}_\alpha \subseteq \mathbb{R}^n_\alpha$.
      i. (Dis)Prove: a subset $\mathcal{U} \subset X$ is open/closed iff $\phi_\alpha(\mathcal{U} \cap \mathcal{U}_\alpha)$ is open/closed in $\mathbb{R}^n$ for each α.
      ii. Prove: all the opens $\tilde{\mathcal{U}}_\alpha \subset \mathbb{R}^n_\alpha$ can be assumed bounded. Namely, for each ϕ_α with a C^r-diffeomorphism $\psi_\alpha \circlearrowright \mathbb{R}^n_\alpha$ (construct it) such that $\psi_\alpha(\tilde{\mathcal{U}}_\alpha) \subset Ball_1(o)$.
   c. Take a C^r-manifold X, for $1 \leq r \leq \infty$, and a closed subset $Z \subset X$. Take a function $f \in C^r(X \setminus Z)$ satisfying: $\lim\limits_{x \to x_o} f^{(k)}|_x = 0$ for each $X \ni x_o \in \partial Z$ and all $0 \leq k \leq r$. (Verify: this condition does not depend on the choice of local coordinates on (X, x_o).) Prove: f extends to a C^r-function on X. When is the extension unique?
   d. Take a partition of unity $\sum \rho_\alpha = 1$ and a compact subset $Z \subseteq X$. Prove: $Z \cap supp(\rho_\alpha) = \varnothing$ except for a finite set of α's.