Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 7.

Submission date: 18.05.2025.

Questions to submit: 1.d. 1.d. 1.f. 2.a. 2.c. 2.d. 3.a.i. 3.a.ii. 3.b. 3.c.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. Below X is a C^r-manifold, $1 \leq r \leq \infty$.
   a. For an arbitrary subset $Z \subset \mathbb{R}^N$ verify: the function $dist(x, Z)$ is continuous.
   b. In the class we had the lemma on refinement of coverings, with the conclusion $\overline{X \cap \mathcal{V}_\alpha}^X \subseteq \mathcal{U}_\alpha$. Does that statement hold with the conclusion $\overline{\mathcal{V}_\alpha} \subseteq \mathcal{U}_\alpha$?
   c. (Dis)prove: for any open covering $\mathbb{R}^n = \cup_{\alpha\in A}\mathcal{U}_\alpha$ there exists a locally finite subcovering, $\mathbb{R}^n = \cup_{\alpha\in A'}\mathcal{U}_\alpha$.
   d. Prove: for any open subset $\mathcal{U}\subset X$ there exists a "bump" function $\rho\in C^r(X)$ satisfying: $\rho|_{\mathcal{U}}>0$, $\rho|_{X\setminus\mathcal{U}}=0$.
   e. Take closed subsets $Z_1, Z_2 \subset X$, such that $Z_1 \cap Z_2 = \varnothing$. Prove: there exists a function $f \in C^r(X)$ satisfying: $f|_{Z_1} = 0$, $f|_{Z_2} = 1$, $0 < f|_{X\setminus(Z_1\cup Z_2)} < 1$.
   f. Suppose two closed subsets are separated by opens, i.e. $Z_i\subset\mathcal{U}_i\subset X$, with $\mathcal{U}_1\cap\mathcal{U}_2=\varnothing$ and $Z_i\cap\partial\mathcal{U}_i=\varnothing$. Prove: there exists $f\in C^r(X)$ satisfying $f|_{Z_1}=0$, $0<f_{\mathcal{U}_1\setminus Z_1}<1$, $f|_{X\setminus(\mathcal{U}_1\cup\mathcal{U}_2)}=1$, $1<f|_{\mathcal{U}_2\setminus Z_2}<2$, $f|_{Z_2}=2$.

2. a. Given an injective C^r-map $X \overset{\phi}{\to} Y$ and a function $Y \overset{f}{\to} \mathbb{R}$, (dis)prove: $f \in C^r(Y)$ iff $\phi^*(f) \in C^r(X)$.
   b. Suppose a set-theoretic map $X \overset{\phi}{\to} Y$ satisfies: $\phi^*(f) \in C^r(X)$ for each $f \in C^r(Y)$. (Dis)Prove: ϕ is a C^r-map of manifolds.
   c. Take a parametrization of a manifold, $(\mathbb{R}^n, o) \overset{\phi}{\rightsquigarrow} (X, x_o) \subset (\mathbb{R}^N, x_o)$. Take the standard inner product on $\mathbb{R}^N_v$ and restrict it to $T_{(X,x_o)}$. Given vectors $u, v \in T_{(\mathbb{R}^n,o)}$, compute the angle between the vectors $d\phi|_o(u)$, $d\phi|_o(v)$. What do you get for $u = \hat{e}_i$, $v = \hat{e}_j$, vectors of the standard basis?
   d. Vector fields ξ, η on $S^2\setminus\{\pm\hat{z}\}\subset\mathbb{R}^3$ are of unit length, go along the parallels/meridians, and ξ is north-oriented, while η is west-oriented. Write the formulas for ξ,η in polar coordinates.

3. a. Compute the integrals: i. $\int\limits_{\{x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\}} (x^{\frac{4}{3}} + y^{\frac{4}{3}})dC.$ ii. $\int\limits_{(x^2+y^2)^2=a^2(x^2-y^2)} |y|dC.$
   b. For which values of $s > 0$ the planar curve defined by $r(\theta) = \frac{1}{1+\theta^s}$, $\theta \in [0, \infty)$ has a finite length?
   c. A curve $C \subset \mathbb{R}^3$ is defined via parameterization in polar coordinates by equations $r = r(\phi)$, $\theta = \theta(\phi)$, for $\phi \in [\phi_0, \phi_1]$. (Here θ is the angle with $\hat{z}$-axis.) Prove: $\int_C f\cdot dC = \int_{\phi_0}^{\phi_1} f\cdot\sqrt{(\partial_\phi r)^2 + r^2(\partial_\phi\theta)^2 + r^2 sin^2(\theta)}d\phi$.
   d. Suppose $\int_X f \cdot dX$ exists. Prove: $|\int_X f \cdot dX| \leq \int_X |f| \cdot dX$.
   e. For $f, g \in C^1(X)$ and $a, b \in \mathbb{R}$ verify: $\int_X(a \cdot f + b \cdot g)dX = \cdots$
   f. Take a manifold admitting a parametrization, $\mathbb{R}^n \supseteq \mathcal{U} \rightsquigarrow X \subset \mathbb{R}^N$. We used this parametrization in the definition of the integral $\int_X fdX$. Verify: the value of the integral does not depend on the choice of parametrization.

4. We have defined "sets of dim$\leq n - 1$."
   a. What does this mean for $n = 1$?
   b. Verify: the class of sets of dim$\leq n-1$ is closed under (arbitrary) intersections and locally finite unions.
   c. Prove: any subset $Z \subset \mathbb{R}^n$ of dim$\leq n - 1$ has Lebesgue measure 0.