Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 8.

Submission date: 25.05.2025.

Questions to submit: 3.a. 3.b. 3.c. 4.a. 4.b. 4.d. 4.e.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. [Catch-up of Calculus.3] We study the volume of the n-dimensional ball, $Ball^{(n)}_R(0) \subset \mathbb{R}^n$.
   a. Take the projection $Ball^{(n)}_R(0) \stackrel{\pi}{\to} Ball^{(2)}_R(0)$, $\pi(\underline{x}) = (x_{n-1}, x_n)$. For each $(x_{n-1}, x_n) \in Ball^{(2)}_R(0)$ verify: $\pi^{-1}(x_{n-1}, x_n) = Ball^{(n-2)}_{\sqrt{R^2 - x^2_{n-1} - x^2_n}}(0) \times \{(x_{n-1}, x_n)\}$.
   b. Obtain the recursion $vol_n Ball^{(n)}_R(0) = \frac{2\pi R^2}{n} \cdot vol_{n-2} Ball^{(n-2)}_R(0)$. Obtain the formula for $vol_n Ball^{(n)}_R(0)$.
   c. Compute $\lim\limits_{n\to\infty} \frac{vol_n Ball^{(n)}_R(0)}{R^n}$ and $\lim\limits_{n\to\infty} \frac{vol_n Ball^{(n)}_R(0) - vol_n Ball^{(n)}_{R-\epsilon}(0)}{vol_n Ball^{(n)}_R(0)}$, for fixed $R > \epsilon > 0$. Interpret the result of the second limit, "For $n \gg 1$ most of the volume is located at …"

2. [Catch-up of Calculus.3]
   a. Compute vol_n of the sets: i. $\{(x, y) | \ (x^2 + y^2)^2 \le 2a^2(x^2 - y^2)\} \subset \mathbb{R}^2$
      ii. $\{(x, y, z) | \ |x + 2y + 3z| + |2x + 3y + z| + |3x + y + 2z| \le 1\} \subset \mathbb{R}^3$.
      iii. $\{\underline{x} | \ \sum_{i=1}^k x_i^2 \le 1 \ge \sum_{i=k+1}^n x_i^2\} \subset \mathbb{R}^n$.
   b. Compute $\iiint_V zye^{x+y^2} dxdydz$, where $V = \left\{0 \le z \le 2, \ \frac{x}{3} \le z \le \frac{x}{2}, \ \frac{y^2}{4} \le z \le y^2\right\} \subset \mathbb{R}^3$.
   c. ("By the symmetry reasons…") Suppose f is integrable on a set $S \subset \mathbb{R}^n$ and both f, S are invariant under the reflections $\{x_j \to -x_j\}$, for all j. Prove: $\int_S f d^n \underline{x} = 2^n \cdot \int_{S \cap \{x_1, \dots, x_n > 0\}} f d^n \underline{x}$.
   d. A set $S \subset \{y > 0\} \subset \mathbb{R}^2_{yz}$ admits vol_2. The body V is obtained by the rotation of S around $\hat{z}$-axis. Compute $vol_3(V)$.
      i. In particular, compute the volume bounded by the standard torus.
      ii. Re-compute this by introducing the toric coordinates, parametrizing the torus.
   e. For which α does $\int\limits_{0 \le y \le x^\alpha \le 1} \frac{dxdy}{x^2 + y^2}$ converge?
   f. Compute $\int_{\mathbb{R}^2} e^{-x^2 - y^2} dxdy$. Deduce: $\int_{-\infty}^{\infty} e^{-x^2} dx = \cdots$.

3. a. Prove the linear algebra identity: $det[\mathbb{1} + \{v_i \cdot v_j\}_{ij}] = 1 + \sum v_i^2$.
   b. Given a function $[0, 1] \stackrel{f \in C^1}{\to} \mathbb{R}^n$, compute the length of its graph, $\Gamma_f \subset [0, 1] \times \mathbb{R}^n$.
   c. For $X \subset \mathbb{R}^3$ with a parametrization $\underline{x}(s, t)$ verify: $\int_X f dX = \int f(\underline{x}(s, t)) \cdot \|\partial_s \underline{x} \times \partial_t \underline{x}\| \cdot dsdt$. (Here $\partial_s \underline{x} \times \partial_t \underline{x}$ is the vector product.)
   d. Take a function $g \in C^1(\mathcal{U})$ and its graph $X := \Gamma_g \subset \mathcal{U} \times \mathbb{R}^1$. Prove: $\int_X f dX = \int_{\mathcal{U}} f \cdot \sqrt{1 + \|\nabla g\|^2} \cdot dx_1 \cdots dx_n$.
   e. Take a hyperplane $L \subset \mathbb{R}^{n+1}$ and a submanifold $X \subset L$, with $k := dim(X) \le n$. Denote the angle between L and $\mathbb{R}^n_{x_1 \dots x_n}$ by α. Take the orthogonal projection $\mathbb{R}^{n+1} \stackrel{\pi}{\to} \mathbb{R}^n_{x_1 \dots x_n}$. Prove: $vol_k(X) \cdot cos(\alpha) = vol_k \pi(X)$.

4. a. The hypersurface $X \subset \mathbb{R}^{n+1}$ is defined by the equation $h(x) = 0$. Suppose the projection $X \stackrel{\pi}{\to} \mathbb{R}^n_{x_1 \dots x_n}$ is injective and $\partial_{x_{n+1}} h$ has no zeros on $X \setminus Z$, where $dim(Z) \le n-1$. Prove: $\int_X f \cdot dX = \int_{\pi(X)} f \frac{\|grad(h)\|}{|\partial_{x_{n+1}} h|} \cdot d\pi(X)$.
   b. Take the standard parametrization of the (standard) torus, $S^1_{\phi_1} \times S^1_{\phi_2} \to X \subset \mathbb{R}^3$. Prove: $\int_X f \cdot dX = \iint_{(\phi_1, \phi_2) \in \mathcal{U}} f(\underline{x}(\phi_1, \phi_2)) \cdot r \cdot (R + r \sin(\phi_1)) d\phi_1 d\phi_2$. In particular, compute the surface area of the torus.
   c. Find the area of the part of the torus that satisfies $x^2 + y^2 \ge a^2$, where a satisfies $R - r < a < R + r$.
   d. Compute the area of the surface $S = \{(x, y, z) | x^2 + y^2 + z^2 = a^2, \ \frac{x^2}{a^2} + \frac{z^2}{b^2} \le 1\}$, here $0 < b \le a$.
   e. Suppose the integrals $\int_X f dX$ and $\int_Y g dY$ exist. Compute $\int_{X \times Y} f(x) \cdot g(y) d(X \times Y)$.
   f. Compute the area of the torus $S^1 \times S^1 \subset \mathbb{R}^4$.