Geometric Calculus 2

201.1.1041 Spring 2025 (D.Kerner)

Homework 9.

Submission date: 2.06.2025.

Questions to submit: 1.e. 1.f. 2.a. 2.c.ii. 3.a. 3.d. 4.a.ii. 4.b. 4.d. 4.e.
(Either typed or in readable handwriting and scanned in readable resolution.)

1. Fix a germ of manifold $(X,x_o)\subset(\mathbb{R}^N,x_o)$. Its normal space is $\mathcal{N}_{(X,x_o)}:=(T_{(X,x_o)})^\perp\subset T_{(\mathbb{R}^N,x_o)}$, the orthogonal complement for the standard inner product on $T_{(\mathbb{R}^N,x_o)}=\mathbb{R}^N_v$.
   a. For a hypersurface-germ $(X,x_o)=\{f(x)=0\}\subset(\mathbb{R}^{n+1},x_o)$, with $\nabla f|_{x_o}\neq 0$, prove: $\mathcal{N}_{(X,x_o)}=Span(\nabla f|_{x_o})$.
   b. Extend a. to the case of several equations.
   c. (Dis)prove: $dim[\mathcal{N}_{(X,x_o)}]$ depends only on the diffeomorphism type of (X,x_o), and not on the embedding $(X,x_o)\subset(\mathbb{R}^N,x_o)$.
   d. (Dis)prove: every map $(X,x_o)\stackrel{\phi}{\to}(Y,y_o)$ induces either $\mathcal{N}_{(X,x_o)}\stackrel{\phi_*}{\to}\mathcal{N}_{(Y,y_o)}$ or $\mathcal{N}_{(X,x_o)}\stackrel{\phi^*}{\leftarrow}\mathcal{N}_{(Y,y_o)}$.
   e. The decomposition $T_{(X,x_o)}\oplus\mathcal{N}_{(X,x_o)}=T_{(\mathbb{R}^N,x_o)}$ implies $T^*_{(X,x_o)}\oplus\mathcal{N}^*_{(X,x_o)}=T^*_{(\mathbb{R}^N,x_o)}=(\mathbb{R}^N_v)^*$.
   Using it verify: $T^*_{(X,x_o)}=\{l\in(\mathbb{R}^N_v)^*|\ l(\mathcal{N}_{(X,x_o)})=0\}$. And accordingly: $T^*X=\{\dots ??\dots\}\subset T^*\mathbb{R}^N$.
   f. Prove: any element of $T^*_{(X,x_o)}$ is presentable as $df|_{x_o}$ for a function $(X,x_o)\stackrel{f}{\to}\mathbb{R}^1$.
   g. (Dis)prove: every map $(X,x_o)\stackrel{\phi}{\to}(Y,y_o)$ induces $T^*X\stackrel{\phi^*}{\leftarrow}T^*Y$.

2. a. Express the forms $dx,dy,dz\in\Omega^1(\mathbb{R}^3)$ in polar coordinates. Express the forms dr, $d\theta$, $d\phi$ in cartesian coordinates.
   b. Take the standard Torus$\subset\mathbb{R}^3$. Take $\omega=\sum\omega_i dx_i\in\Omega^1(\mathbb{R}^3)$. Present the restriction $\omega|_{Torus}$ in the coordinates (ϕ,ψ) on this torus.
   c. Take a map of manifolds $X\stackrel{\phi}{\to}Y$ and the corresponding pullback $\Omega^1(X)\stackrel{\phi^*}{\leftarrow}\Omega^1(Y)$.
      i. Verify: $\phi^*(f\cdot\omega)=\phi^*(f)\cdot\phi^*(\omega)$.
      ii. Verify: $\phi^*df=d\phi^*f$. In particular, for a submanifold $X\subseteq Y$ one has $(df)|_X=d(f|_X)$.
      iii. For two maps $X\stackrel{\phi}{\to}Y\stackrel{\psi}{\to}Z$ verify the chain rule: $T^*_{(X,x_o)}\stackrel{(\psi\circ\phi)^*=\phi^*\circ\psi^*}{\longleftarrow}T^*_{(Z,Z_o)}$ and $\Omega^1(X)\stackrel{(\psi\circ\phi)^*=\phi^*\circ\psi^*}{\longleftarrow}\Omega^1(Z)$.

3. a. Take a function $f\in C^r(\mathcal{U})$ for an open $\mathcal{U}\subseteq\mathbb{R}^n$. Define $\omega=df$ and $\xi=grad(f)$. Compute $\omega(\xi)$.
   b. Verify: the pairing of 1-forms and vector fields, $(\omega,\xi)\to\omega(\xi)$, is $C^r(X)$-bilinear, i.e.
   $\omega(\xi_1+\xi_2)=\dots,\qquad \omega(f\cdot\xi)=f\cdot(\omega(\xi))=(f\cdot\omega)(\xi),\qquad (\omega_1+\omega_2)\xi=\dots.$
   c. We have defined the map $C^r(X)\to\Omega^1(X)$, $f\to df$. Verify: it is $\mathbb{R}$-linear, and satisfies the Leibniz rule.
   d. A submanifold $X\subset\mathbb{R}^{n+1}$ is defined by the equation $f(\underline{x})=0$ for some $f\in C^r(\mathbb{R}^N)$. Compute $df|_X$.

4. a. Compute $\int_C\vec{F}\cdot d\vec{C}$ in the following cases:
      i. $\vec{F}(\underline{x})=(x_1,x_2^2,\dots,x_n^n)$, $\vec{C}=\{(sin(t),sin^2(t),\dots,sin^n(t))|\ t\in[0,\pi]\}$.
      ii. $\int\limits_{\left\{\substack{x^2+y^2+z^2=1,\\ y=-x,\quad z>0}\right\}}(z^2dx+3y^2dy-x^2dz)$, the curve begins at $(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0)$ and ends at $(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$.
   b. Given a smooth curve $C\subset\mathbb{R}^n$, take the projections onto the coordinate axes $\{C\stackrel{\pi_i}{\to}Span(\hat{x}_i)\}$. Suppose these are diffeomorphisms onto their images. Prove: $\int_C\vec{F}\cdot\vec{C}=\sum\int_{\pi_i(C)}F_i(\pi_i^{-1}(x_i))dx_i$. What are the orientations of $\pi_i(C)$ here?
   c. How to adjust the formula in b. when some projections are not bijective? (e.g. how to convert $\int_{S^1}\vec{F}\cdot d\vec{C}$ into $\int(..)dx+\int(..)dy$?) Using this formula recompute the integrals in part a.
   d. (Newton-Leibniz) Take a parameterized oriented curve $\mathbb{R}^1\supset(a,b)\stackrel{\phi}{\rightsquigarrow}C\subset\mathbb{R}^N$ and $\omega=df\in\Omega^1(C)$. Verify: $\int_{\vec{C}}\omega=f(\phi(b))-f(\phi(a))$.
   e. Take a smooth oriented curve $S^1\cong C\subset\mathbb{R}^N$. Prove: $\int df=0$ for any $f\in C^1(C)$.