Introduction to Riemann Surfaces, 201.2.5101

Moed.A, 24.02.2026, three hours.

(Lecturer: Dmitry Kerner)

No auxiliary material is allowed.

Solve all the questions. The total is 105 points.

Do not write in red color!

Below X is a Riemann surface. In the compact case you can freely use the Riemann-Roch theorem.

1. (15 points) Take a conic $X \subset \mathbb{P}^2$ and two distinct lines $L_j = \{l_j(\underline{z}) = 0\} \subset \mathbb{P}^2$ satisfying: $L_1 \cap L_2 \cap X$ =one point. Prove: the function $\frac{l_1}{l_2}|_X \in M(X)$ has just one zero and one pole.

Prove: $X \approx \mathbb{P}^1$.

2. (15 points) Prove: any compact Riemann surface of genus zero is isomorphic to $\mathbb{P}^1$.

3. (15 points) Let X be a compact Riemann surface of genus g. Prove: $dim_{\mathbb{C}} \Omega^1(X) = g$.

4. (15 points) For X-compact, a point $p \in X$, and any $\omega \in H^0(\Omega_X(n \cdot p))$ prove: $Res_p \omega = 0$.

5. (15 points) Prove: the vector space of analytic vector fields on $\mathbb{P}^1$ is of $dim_{\mathbb{C}}$ =3.

6. (15 points) Given meromorphic functions $f_1, \dots, f_N \in M(X)$, take their total set of poles, $S = supp[\sum_j div_\infty(f_j)] \subset X$. Prove: the map $X \setminus S \stackrel{(f_1,\dots,f_N)}{\to} \mathbb{C}^N$ extends to an analytic map $X \to \mathbb{P}^N$.

7. (15 points) Let $X = \{y^2 = \prod_{j=1}^d (x - x_j)\} \subset \mathbb{C}^2$, for d-odd. Take the corresponding hyperelliptic curve, with the map $\tilde{X} \stackrel{\phi}{\to} \mathbb{P}^2$, whose image is $\bar{X}$.

Compute $div(\phi^* dx) \in Div(\tilde{X})$ and the residues of $\phi^* dx$ at all the points of $\tilde{X}$.

Good Luck!