Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 0. Not for submission

It is important to (fully) solve this homework before the first lecture.

Below $\Bbbk$ is one of $\mathbb{R}, \mathbb{C}$. A (real or complex) plane algebraic curve is the subset $C = \{p(x,y)=0\} \subset \Bbbk^2$, where $p(x,y) \in \Bbbk[x,y]$. The degree of the curve is defined as the degree of the polynomial $p(x,y.)$ For example, $deg(x^4+x^2y^3+y-1)=5$.

1. a. Prove: a locally constant function on a connected topological space is constant.

b. Take a function $\mathbb{R}^n \overset{f\in C^\infty}{\to} \mathbb{R}^m$, with $f(o)=o$. Establish the normal forms:

i. If $n \geq m$ and $rank[f'|_o]=m$, then for some local (C^∞) coordinates at $o \in \mathbb{R}^m$ the function is $f(\underline{x})=(x_1,\dots,x_m)$.

ii. If $n \leq m$ and $rank[f'|_o]=n$, then for some local (C^∞) coordinates at $o \in \mathbb{R}^n$ the function is $f(\underline{x})=(x_1,\dots,x_n,0,\dots,0)$.

c. (Complex hyperbola) Prove: the subset $X=\{z\cdot w=1\}\subset\mathbb{C}^2$ is a C^∞-manifold, and $X \overset{C^\infty}{\underset{\sim}{\longrightarrow}} \mathbb{C}\setminus o$.

d. Describe (geometrically) the subset $\{z^4=w^4\}\subset\mathbb{C}^2$. [This is not a "precise question".]

e. At which points is the following curve (non)smooth? $\{y^2=(x^2-1)^3x\}\subset\mathbb{R}^2$?

2. The action of linear transformations, $GL(2,\Bbbk) \circlearrowright \Bbbk^2$, is defined by $(U,(x,y)) \to (x,y)\cdot U$. The action of parallel translations, $\Bbbk^2 \circlearrowright \Bbbk^2$, is defined by $(\vec{v},(x,y)) \to (x+v_x, y+v_y)$.

a. (Lines in the plane, i.e. curves of degree=1)

i. Prove: any line in $\Bbbk^2$ can be brought, by the action of $\Bbbk^2\times GL(2,\Bbbk)$, to the line $\{y=0\}$.

ii. Try to visualize/imagine the line $\{a_xx+a_yy=1\}\subset\mathbb{C}^2$.

b. (Plane conics, i.e. curves of degree=2)

i. Let $C=\{x^2+y^2=1\}\subset\Bbbk^2$. Take the projection onto the $\hat{x}$-axis $C \overset{\pi_x}{\to} \Bbbk^1$. Describe the image, $\pi_x(C)$. How many preimages has a point $x\in\pi_x(C)$? Do this both for $\Bbbk=\mathbb{R}$ and $\Bbbk=\mathbb{C}$.

ii. Let $p(x,y)\in\mathbb{R}[x,y]$ be of degree 2. Prove that by linear transformations (the group $GL(2,\mathbb{R})$), the parallel translations (the group $\mathbb{R}^2$), and the scaling ($p(x,y)\to\lambda\cdot p(x,y)$, for some $\lambda\in\mathbb{R}\setminus\{0\}$) the curve $\{p(x,y)=0\}\subset\mathbb{R}^2$ can be brought to one (and only one) of the following forms: $y^2\pm x^2=0$, $y^2\pm x^2=1$, $x^2+y^2=-1$, $y=x^2$, $x^2=1$, $x^2=0$. Draw the corresponding curves. (These are called: canonical forms of plane conics.) Which of the curves are smooth?

iii. Let $p(x,y)\in\mathbb{C}[x,y]$ be of degree 2. What are the canonical forms in this case?

iv. Realize the real curves $\{x^2+y^2=1\}$, $\{x^2-y^2=1\}$, $\{-x^2+y^2=1\}\subset\mathbb{R}^2$ as sections of the complex curve, $\{x^2+y^2=1\}\subset\mathbb{C}^2$. (e.g. by $Im(x)=0=Im(y)$) Try to imagine how the complex curve is located in $\mathbb{C}^2$.

c. (Plane cubics) Trace the change of the curve $\{y^2=x^3+x^2+\epsilon\}\subset\mathbb{R}^2$ for $\epsilon\in(-1,1)$. (You can use the computer.) What happens at $\epsilon=0$? Try to imagine the complex curve $\{y^2=x^3+x^2+\epsilon\}\subset\mathbb{C}^2$. (Google: "plane cubic", "nodal cubic")

Trace the change of the curve $\{y^2=x^3+\epsilon\cdot x^2\}\subset\mathbb{R}^2$ for $\epsilon\in(-1,1)$.

3. a. Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space, with the basis $\langle 1,i\rangle$. Write down the presentation matrix of the operator $z\to i\cdot z$ in this basis.

b. For $f\in\mathcal{O}(\mathcal{U})$ verify: the Cauchy-Riemann conditions can be written as $\partial_y f = i\cdot\partial_x f$.

c. Take a function $\mathbb{C}\overset{f}{\to}\mathbb{C}$ analytic at $o\in\mathbb{C}$. Take its real presentation, $\mathbb{R}^2\overset{f_\mathbb{R}}{\to}\mathbb{R}^2$.

i. Take the matrix of (real) partial derivatives $[f_\mathbb{R}]'|_o\in Mat_{2\times2}(\mathbb{R})$. Verify: $[f_\mathbb{R}]'|_o$ is presentable as $\alpha\cdot\mathbb{1}+A$, for some $\alpha\in\mathbb{R}$ and $A\in Mat^{anti-symmetric}_{2\times2}(\mathbb{R})$.

ii. Prove: $det([f_\mathbb{R}]'|_o)=|f'|_o|^2$ (the complex derivative). In particular, $det([f_\mathbb{R}]'|_o)=0$ iff $f'|_o=0$.

4. a. Prove: the series $\sum a_j x^j \in \Bbbk[\![x]\!]$ converges on $Ball_r(o)$ iff $|a_j| < \frac{C}{r^j}$ for some $C \in \mathbb{R}$ (and all $j \in \mathbb{N}$).
b. Find the radius of convergence of the Taylor series of $f(z) = tan(i \cdot sin(z))$ at $z = o$.
c. Suppose the series $\sum a_j x^j$ converges on $(-r, r)$. Prove: $\frac{d}{dx}(\sum a_j x^j) = \sum \frac{d}{dx}(a_j x^j)$, and the new series also converges on $(-r, r)$.

5. a. Suppose $f \in \mathcal{O}(\mathbb{C})$ is a bi-periodic function, i.e. $f(z + w_1) = f(z) = f(z + w_2)$ for $z \in \mathbb{C}$. [Here the periods satisfy $\frac{w_1}{w_2} \notin \mathbb{R}, \pm\infty$.] Prove: $f = const$.
b. Let $f \in \mathcal{O}(\mathbb{C})$, suppose its image is not dense, i.e. $\overline{f(\mathbb{C})} \subsetneq \mathbb{C}$. Prove: $f = const$.
c. Find $max_{z \in [0,2\pi]^2} |sin(z)|$.
d. Does there exist $f \in \mathcal{O}(Ball_\epsilon)$ satisfying $|f(z)| = e^{|z|}$?
e. Prove: $f \in \mathcal{O}(\mathcal{U})$ has at most a finite number of zeros in each compact subset of $\mathcal{U}$.
f. Let $f \in \mathcal{O}(Ball_2(o))$ and suppose $\oint_{|z|=1} \frac{f(z) \cdot dz}{(n+1)z - 1} = 0$ holds for all $n \in \mathbb{N}$. Does this imply $f(z) \equiv 0$?
g. Suppose a meromorphic function f has only finite number of poles on $\mathbb{C}$ and at most a pole at infinity. Prove: f is the ratio of two polynomials.

6. a. Find the order of $f(z) = z^2 \cdot Log(1 + z^5) - sin(z^7)$ at $z = 0$, i.e. $ord_o f(z)$. Here Log is the principal branch.
b. Let $f \in \mathcal{O}(Ball_\epsilon)$ such that $ord_o f = \infty$. Prove: $f = 0$.
c. Suppose $ord_{z_o} f = p$. Prove: $f(z) = (z - z_o)^p \cdot g(z)$, where $g \in \mathcal{O}$ and $g(z_o) \neq 0$.
d. Take a Laurent power series $\sum_{n \in \mathbb{Z}} c_n z^n$. Express its radii of convergence (r, R) via $\{c_\bullet\}$.
e. Find and classify all the singular points of $f(z) = \frac{tan(z\sqrt{2})}{cotan(z\sqrt{3})}$. At each such point specify $ord_{z_o} f$.
f. Expand the function $\frac{sin \frac{1}{z}}{z-2}$ into Laurent series in all the annuli in which the function is defined.
g. Suppose $f \in \mathcal{O}(Ball_\epsilon \setminus \{o\})$ satisfies: $\lim_{z \to o} f(z) \cdot z = 0$. Prove: o is a removable singularity of f.
h. Does there exist a function $f \in \mathcal{O}(Ball_\epsilon \setminus o)$ satisfying: $|f(z)| \geq C \cdot e^{\frac{1}{|z|}}$?

7. a. (The local normal form) Let $f \in \mathcal{O}(Ball_\epsilon(z_o))$ and denote $p = ord_{z_o}(f(z) - f(z_o))$. Prove: there exist local coordinates at z_o for which $f(w) = f(z_o) + w^p$. [You can use the $\mathcal{O}$-implicit function theorem] Extend this statement to functions with poles at z_o.
b. Take $f(z)$ from 6.a. Prove: for any $1 \gg \epsilon > 0$ there exists $\delta > 0$ such that the equation $f(z) = w_o$ (for $w_o \in Ball_\delta(o)$) has exactly $ord_o f(z)$ solutions in $Ball_\epsilon(o)$.
c. Let $const \neq f \in \mathcal{O}(\mathcal{U})$ for $\mathcal{U} \subseteq \mathbb{C}$ path-connected. Prove: $f(\mathcal{U}) \subset \mathbb{C}$ is an open subset.
d. Take $f \in \mathcal{O}(Ball_\epsilon(o))$. Prove: f is locally (analytically) invertible at o iff f is locally injective. Does this statement hold for real-analytic functions?

8. Take a topological space X with equivalence relation $\sim$, and the set $X/_\sim$ with the quotient topology.
a. Take $S^1 \subset \mathbb{R}^2$ with the equivalence generated by reflections $(x, y) \sim (-x, -y)$. Prove: $S^1/_\sim \overset{C^0}{\approx} S^1$.
b. Take $\mathbb{R}^2$ with the equivalence relation $[(x_1, y_1) \sim (x_2, y_2)$ if $x_1 - x_2 \in \mathbb{Z} \ni y_1 - y_2$.] Prove: $\mathbb{R}^2/_\sim \overset{C^0}{\approx} S^1 \times S^1$.
c. Prove: if X is compact, then so is $X/_\sim$.
d. Take $\mathbb{R}$ with the equivalence relation $[x_1 \sim x_2$ if $0 \neq \frac{x_1}{x_2} \neq \infty]$. Describe the topological space $\mathbb{R}/_\sim$. Verify: it is non-Hausdorff.

9. a. Prove: $S^n \subset \mathbb{R}^{n+1}$ is oriented.
b. Take a manifold with two charts $X = \mathcal{U}_1 \cup \mathcal{U}_2$, suppose $\mathcal{U}_1 \cap \mathcal{U}_2$ is path-connected. Prove: X is orientable.
c. Let X be orientable with d connected components. How many different orientations it has?
d. Express the restriction $dy \wedge dz|_{S^2}$ in polar coordinates on $S^2 \subset \mathbb{R}^3$.