Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 1.

Submission date: 8.11.2025.

Questions to submit: 1.c. 1.d. 2.a. 2.f. 3.b.i. 3.b.ii. 4.a.

(Either typed or in readable handwriting and scanned in readable resolution.)

1.a. We have defined the germs of sets and functions via equivalence relations. Verify: these are indeed equivalence relations.

b. Define the basic operations on a *finite* number of germs,

$$\bigcap_i (X_i,o) := (\bigcap_i X_i, o), \qquad \bigcup_i (X_,o) := (\bigcup_i X_i, o), \qquad (X,o)\setminus(Y,o) := (X\setminus Y, o), \qquad \prod_i (X_i,0) := (\prod_i X_i, o).$$

Verify: the results are well defined, i.e. do not depend on the choice of representatives.

c. Show (by examples) that infinite intersections, unions, products of germs are not well defined.

d. (The germ of the image vs the image of the germ) Define the function $(\mathbb{R}^2,o)\xrightarrow{f}(\mathbb{R}^2,o)$ by $f(x,y)=(x,x\cdot y)$. Prove: the germ $(f(Ball_\epsilon(o)),o)$ depends on ϵ. Thus the image $f(\mathbb{R}^2,o)$ is not well defined. In particular, $f(\mathbb{R}^2,o)\neq(f(Ball_\epsilon(o)),o)$.

e. Define the *set of convergence* of a power series, $S:=\{\underline{z}|\ \sum a_{\underline{m}}\underline{z}^{\underline{m}} \text{ converges}\}\subseteq\mathbb{C}^n$.

i. Recall that for $n=1$ one has $Disc_R(0)\subseteq S\subseteq\overline{Disc_R(0)}$, where R is the radius of convergence. Does this hold for $n>1$ and polydiscs?

ii. Disprove: $S\subseteq\overline{Int(S)}$ and S is path-connected. (Hint: $f(z_1,z_2)=\frac{z_1}{1-z_2}$ and $f(z_1,z_2)=\frac{1}{1-z_1z_2}$)

2.a. Consider $\mathbb{C}^n\supseteq\mathcal{U}\xrightarrow{f}\mathbb{C}$ as a real valued function, $\mathbb{R}^{2n}\supseteq\mathcal{U}\xrightarrow{[f]}\mathbb{R}^2$, with $z_j=x_j+i\cdot y_j$. Define (formally): $\frac{\partial[f]}{\partial z_j}:=\frac{\partial_{x_j}[f]-\partial_{y_j}[i\cdot f]}{2}$ and $\frac{\partial[f]}{\partial\bar{z}_j}:=\frac{\partial_{x_j}[f]+\partial_{y_j}[i\cdot f]}{2}$.

Prove: f is $\mathbb{C}$-differentiable iff $[f]$ is $\mathbb{R}$-differentiable and $\{\frac{\partial[f]}{\partial\bar{z}_j}=0\}_{j=1,\dots,n}$.

b. Take a connected open subset $\mathcal{U}\subseteq\mathbb{C}^n$. Prove: $\mathcal{O}(\mathcal{U})$ is a unital commutative ring with no zero divisors. (Namely, if $f\cdot g=0$ then $f=0$ or $g=0$.)

c. (Chain rule) Given analytic functions $\mathcal{U}\xrightarrow{f}\mathcal{V}\xrightarrow{g}\mathbb{C}$ prove: $g\circ f\in\mathcal{O}(\mathcal{U})$ and $(g\circ f)'=g'|_f\cdot f'$.

d. Prove Liouville's theorem: if $f\in\mathcal{O}(\mathbb{C}^n)$ is bounded, then it is constant.

e. The assumption of the uniqueness theorem for $f\in\mathcal{O}(\mathbb{C}^n)$ with $n\geq2$ is much stronger than that for $f\in\mathcal{O}(\mathbb{C}^1)$. Show by example(s) that the assumption of $n=1$ is not sufficient for $n\geq2$.

f. Let $\mathbb{R}^n\subset\mathbb{C}^n$ be the real part, defined by $\{Im(z_j)=0\}_j$. For $f,g\in\mathcal{O}(\mathbb{C}^n,o)$ prove: if $f=g$ in a real neighborhood of o (i.e. in some $Ball_\epsilon(o)\cap\mathbb{R}^n$) then $f|_{(\mathbb{C}^n,o)}=g|_{(\mathbb{C}^n,o)}$.

3.a. We have proved the $\mathcal{O}$-implicit function theorem for the case of one equation. Prove the general version.

b. Take a function-germ $(\mathbb{C}^n,o)\xrightarrow{f\in\mathcal{O}}(\mathbb{C}^m,o)$. Establish the normal forms:

i. If $n\geq m$ and $rank[f'|_o]=m$, then for some $\mathcal{O}$-coordinates on $(\mathbb{C}^m,o)$ the function is $f(\underline{x})=(x_1,\dots,x_m)$.

ii. If $n\leq m$ and $rank[f'|_o]=n$, then for some $\mathcal{O}$-coordinates on $(\mathbb{C}^n,o)$ the function is $f(\underline{x})=(x_1,\dots,x_n,0,\dots,0)$.

c. Deduce the open mapping theorem for $\mathbb{C}^n\supseteq\mathcal{U}\xrightarrow{f\in\mathcal{O}}\mathbb{C}^m$ with $n\geq m$: if $rank[f']=n$ everywhere on $\mathcal{U}$, then f sends open sets to open sets.

4. Let $X=\{p(\underline{z})=0\}\subset\mathbb{C}^n$ for a polynomial $const\neq p(\underline{z})\in\mathbb{C}[\underline{z}]$.

a. Prove: either $p(\underline{z})$ does not depend on z_n, or the image of the projection $X\xrightarrow{\pi_n}\mathbb{C}^{n-1}_{z_1\dots z_{n-1}}$ is dense. Deduce: X is never compact.

b. Suppose for every point $z^o\in\mathbb{C}^{n-1}_{z_1\dots z_{n-1}}$ the number of preimages (in X) is finite. Prove: this number is at most $deg(p)$.