Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 10.

Submission date: 16.01.2026.

Questions to submit: 2.f. 3.b. 3.d. 4.b. 4.c. 4.g.

(Either typed or in readable handwriting and scanned in readable resolution.)

Below X is a compact Riemann surface.

1. a. Take smooth curves $X, Y \subset \mathbb{P}^2$ of $deg = d$, so that $\sharp(X \cap Y) = d^2$. Suppose $d \cdot e$ points of $X \cap Y$ lie on a smooth curve $Z \subset \mathbb{P}^2$ of $deg = e$. Prove: the remaining $d(d-e)$ points lie on a curve of $deg = (d-e)$. Hint: for some $[c_x : c_y] \in \mathbb{P}^1$ the curve "$c_X \cdot X + c_Y \cdot Y$" intersects Z in "too many points".

b. Suppose two cubics in $\mathbb{P}^2$ intersect at the distinct points $p_1, \dots, p_9$ and a third cubic passes through $p_1, \dots, p_8$. Prove: it also passes through p_9.

c. Given functions $f_1, \dots, f_N \in M(X)$, take their total set of poles, $S = supp[\sum_j div_\infty(f_j)] \subset X$. Prove: the map $X \setminus S \overset{(f_1,\dots,f_N)}{\to} \mathbb{C}^N$ extends to an analytic map $X \overset{F}{\to} \mathbb{P}^N$.

2. a. Verify: $h^0(\mathcal{O}_X(D)) > 0$ iff $D \sim D' \geq 0$.

b. For $X \overset{f}{\to} \mathbb{P}^1$ consider the divisors $f^*(D)$, for all $D \in Div(\mathbb{P}^1)$ of a fixed $deg = d$.
Verify: they are linearly equivalent. [In particular, $div_o(f) \sim div_\infty(f)$.]

c. Extend b. to the maps $X \overset{f}{\to} Y$ and the pull-backs.

d. And what is the version for a map $X \overset{\phi}{\to} \mathbb{P}^n$?

e. Let $D_1, D_2 \in Div(X)$. Disprove: if $D_1 \sim D_2$ then $D_1 = f^*(o)$ and $D_2 = f^*(\infty)$ for some $X \overset{f}{\to} \mathbb{P}^1$.

f. Prove that 2.e. holds with the additional assumptions: $0 \leq D_1, D_2$ and $supp(D_1) \cap supp(D_2) = \varnothing$.

3. a. Let $D = \sum m_j \cdot p_j \in Div(\mathbb{P}^1)$. Describe $H^0(\Omega^1_{\mathbb{P}^1}(D))$.

b. Fix some $0 \neq f \in M(X)$ and take the map $\Omega^1_{mer}(X) \overset{f\cdot}{\to} \Omega^1_{mer}(X)$. Identify the image of $H^0(\Omega_X(D))$.

c. Fix some $0 \neq \omega \in \Omega^1_{mer}(X)$ and take the map $M(X) \overset{\cdot\omega}{\to} \Omega^1_{mer}(X)$. Identify the image of $H^0(\mathcal{O}_X(D))$.

d. Prove "the two basic properties":

i. If $D \leq D'$ then $dim_{\mathbb{C}} H^0(\mathcal{O}_X(D'))/H^0(\mathcal{O}_X(D)) \leq deg(D' - D)$.

ii. For $deg(D) \geq 0$: $\quad h^0(\mathcal{O}_X(D)) \leq deg(D) + 1$.

Deduce: if $D \leq D'$, then $deg(D) - h^0(\mathcal{O}_X(D)) \leq deg(D') - h^0(\mathcal{O}_X(D'))$.

4. (Below you can use Riemann-Roch)

a. Any compact Riemann surface of $g = 0$ is isomorphic to $\mathbb{P}^1$.

b. If $deg H^0(\mathcal{O}_X(D)) = deg(D) + 1$ for <u>some</u> D with $deg(D) > 0$ then $X \approx \mathbb{P}^1$.

c. $dim_{\mathbb{C}} \Omega^1(X_g) = g$.

d. For any $\omega \in \Omega^1_{mer}(X_{g=1})$, the divisor $div_X(\omega)$ is principal.

e. If $deg(D) > 2g(X) - 2$, then $h^0(\mathcal{O}_X(D)) = deg(D) - g(X) + 1$. What happens for $deg(D) = 2g(X) - 2$?

f. (Dis)prove: if $h^0(\mathcal{O}_X(p_1 + p_2)) = 2$ for some $p_1, p_2 \in X$ then $g(X) = 1$.

g. We have obtained analytic forms on hyperelliptic curves in hwk.7, q.4.d. Verify: they form a basis of $\Omega^1(X)$.