# Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

## Homework 11.

Submission date: 23.01.2026.

Questions to submit: 1.c. 2.c. 3.b. 4.b. 5.b.

(Either typed or in readable handwriting and scanned in readable resolution.)

Below $X$ is a compact Riemann surface.

**1.a.** Let $const \neq f \in H^0(\mathcal{O}_X(m \cdot p))$ for some $m \in \mathbb{N}$ and $p \in X$. Suppose $g(X) \geq 1$. Prove: $f \neq h^m$ for any $h \in M(X)$.

**b.** For any $\omega \in H^0(\Omega_X(n \cdot p))$ prove: $Res_p \omega = 0$.

**c.** $X \subset \mathbb{P}^n$ is called non-degenerate if it does not lie in a hyperplane. Prove: in this case $deg(X) \geq n$.

i. Conclude: if $deg(X) = 2$ then $X \subset \mathbb{P}^2$. ["Conics are inherently planar curves"]

ii. Suppose $X \subset \mathbb{P}^3$ is a non-degenerate cubic. Fix any point $p \in X$ and prove: $h^0(\mathcal{O}_X(p)) = 2$. Deduce: $X \approx \mathbb{P}^1$.

**2.** In the proof of "{all the compact R.S. of genus 1}={all the complex tori}" we made the following statements. Write their proofs.

**a.** The transition functions of the space $\cup_{jk} \mathcal{U}_{jk}$ are analytic.

**b.** The projection $\cup_{jk} \mathcal{U}_{jk} \xrightarrow{\pi} X_{g=1}$ is an (analytic) unramified covering.

**c.** Take a 'constant' form $0 \neq \omega \in \Omega^1(X_{g=1})$. Then the map $\cup_{jk} \mathcal{U}_{jk} \xrightarrow{\phi} \mathbb{C}$, by $p \to \int_{[o \rightsquigarrow p]} \pi^* \omega$, is well defined. And this is a global analytic isomorphism.

**d.** Take the map $\pi_{\mathbb{C}} := \pi \circ \phi^{-1} : \mathbb{C} \to X_{g=1}$. Verify: the subset $\pi_{\mathbb{C}}^{-1}(o) \subset \mathbb{C}$ is a sublattice.

**3.** In the proof of "{all the compact R.S. of genus 1}={all the smooth cubics in $\mathbb{P}^2$}" we made the following statements. Write their proofs.

**a.** Fix $p \in X_{g=1}$. Then $H^0(\mathcal{O}_X(3p) = \mathbb{C}\langle 1, f, f' \rangle$.

**b.** There exists a local coordinate on $(X, p)$ for which $f(z) = \frac{1}{z^2} + z^2 \cdot (\cdots)$.

**c.** This function $f$ satisfies the (global) equation: $(f')^2 = 4(f - x_1) \cdot (f - x_2) \cdot (f - x_3)$.

**d.** The map $X_{g=1} \setminus \{p\} \xrightarrow{(f,f')} \mathbb{C}^2$ extends to the map $X_{g=1} \xrightarrow{[1:f:f']} \mathbb{P}^2$. It it an embedding. And its image is a smooth cubic.

**4.** (Continuing question 3 of hwk. 9) Fix a smooth curve $X = \{f(x,y,z) = 0\} \subset \mathbb{P}^2$ and consider the Hessian matrix, $f'' \in Mat_{3\times3}(\mathbb{C}[x,y,z])$. Assume $deg(f) \geq 2$.

**a.** Prove: the points of $X \cap \{det[f''] = 0\}$ are exactly the flexes of $X$.
Hint: apply $GL(3)$ to set $p = [0:0:1]$, with $T_{(X,p)} = \{y = 0\}$. (How does this affect $det[f''|_p]$?)
Thus we can assume $f(x,y,z) = z^{d-1}y + az^{d-2}x^2 + y^2(\dots) + xy(\dots) + x^3(\dots)$. Compute $det[f''|_p]$.

**b.** Verify: $p \in X$ is an ordinary flex iff $deg(div_{(X,p)} det[f'']) = 1$.
More generally, what is the relation between the local intersection multiplicity of $X$, $\{det[f''] = 0\}$ at $p$, and the multiplicity of the flex of $X$ at $p$?

**c.** Verify: $det[f'']|_X \neq 0$. Conclude: $X$ has exactly $d \cdot 3(d-2)$ flexes, counted with multiplicities.

**5.** (Weierstraß normal form for smooth plane cubics, $\{f(x,y,z) = 0\} \subset \mathbb{P}^2$)

**a.** Prove: a smooth plane cubic has only ordinary flexes. Conclude: a smooth cubic has precisely 9 flexes.

**b.** By $GL(3)$ put one of the flexes to $[0:1:0] \in \mathbb{P}^2$ with the tangent line $\{z = 0\}$. Then $f$ consists of monomials $y^2z$, $yxz$, $x^3$, $x^2z$, $xz^2$, $z^3$. Apply now a $GL(3, \mathbb{k})$ transformations to get rid of $yzx$, $z^3$. Then arrive at the Weierstraß normal form of smooth cubic: $y^2z = x(x-z)(x - \lambda \cdot z)$, with $\lambda \neq 0, 1$.

**6.** We have proved: "Complex tori up to isomorphisms are parameterized by $\mathbb{H}/_{SL(2,\mathbb{Z})}$". (Here $\mathbb{H}$ is the upper half-plane.) Write the proof in all the details.