Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 2.

Submission date: 16.11.2025.

Questions to submit: 1.e. 2.c. 3.b. 4.a. 4.b. 5.b.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Let $f \in \mathcal{O}(\mathbb{C})$ with no critical points. Is $\mathbb{C} \stackrel{f}{\to} f(\mathbb{C}) \subseteq \mathbb{C}$ a parametrization of its image?

b. We have defined the non-embedded $\mathbb{C}$-manifolds. Verify: every $\mathbb{C}$-submanifold $X \subset \mathbb{C}^N$ is a $\mathbb{C}$-manifold.

c. Verify: the composition of analytic maps (of analytic manifolds) is analytic.

d. Take an analytic map $X \stackrel{f}{\to} Y$ and its graph $\Gamma_f = \{(x, f(x))\} \subset X \times Y$. Verify: $\Gamma_f \stackrel{\mathcal{O}}{\rightsquigarrow} X$.

e. Take a $\mathbb{C}$-submanifold $X \subset \mathbb{C}^N$. For any $f \in \mathcal{O}(\mathbb{C}^N)$ and its restriction $f|_X$ verify: $f|_X \in \mathcal{O}(X)$.

2. The following was proved in the class. Write down these proofs in full detail.

a. all complex manifolds are orientable (when considered as real manifolds);

b. $S^2 \subset \mathbb{R}^3$ has a Riemann surface structure;

c. $\mathbb{P}^1_{\mathbb{C}}$ is a compact Riemann surface, and $\mathbb{P}^1_{\mathbb{C}} \stackrel{\mathcal{O}}{\rightsquigarrow} \mathbb{C} \cup \infty$.

d. Fix two homogeneous polynomials of the same degree, $p(z_0, z_1), q(z_0, z_1) \in \mathbb{C}[z_0, z_1]$. Then $\frac{p(z_0,z_1)}{q(z_0,z_1)}$ defines a holomorphic function at all the points of $\mathbb{P}^1_{\mathbb{C}}$ where q does not vanish.

3. Describe the topological space $\mathbb{C}^1/\mathbb{C}^*$ for the action $\lambda \circ (z_1, z_2) = (\lambda z_1, \lambda z_2)$. [Verify: this space is non-Hausdorff.]

In the class we have constructed complex tori $\mathbb{C}/L$.

a. Prove: any complex torus is Hausdorff, second countable, and homeomorphic to $S^1 \times S^1$.

b. Prove: any complex torus is an abelian group. Moreover, this is an abelian group with division, i.e. for any $x \in \mathbb{C}/L$ and $n \in \mathbb{N}$ exists $y \in \mathbb{C}/L$ satisfying: $n \cdot y = x$. How many such y there exist for a fixed pair (x, n)?

c. Prove: the projection $\mathbb{C} \stackrel{\pi}{\to} \mathbb{C}/L$ is a holomorphic map which is locally isomorphism.

4. Take the curve $C = \{y^2 = x^3\} \subset \mathbb{C}^2$. Denote by π the projection from $(0,0)$ to the line $\{x = 1\} \subset \mathbb{C}^2$.

a. Prove: π restricts to the homeomorphism $C \setminus (0,0) \rightsquigarrow \mathbb{C}^1 \setminus \{0\}$. (Write the explicit formulae.) Verify: this extends to the homeomorphism $C \rightsquigarrow \mathbb{C}^1$.

b. Verify: this homeomorphism defines the structure of complex manifold on C, and $C \stackrel{\mathcal{O}}{\rightsquigarrow} \mathbb{C}$. Verify: $C \subset \mathbb{C}^2$ is not a submanifold.

c. Could one use, instead of the point $(0,0)$, some other point of $\mathbb{C}^2$?

5. Below X, Y are Riemann surfaces.

a. Given two *homeomorphisms* $\mathbb{R}^1 \supseteq (a,b) \xrightarrow[\sim]{\phi_1,\phi_2} (c,d) \subseteq \mathbb{R}^1$, suppose $\phi_1 \circ \phi_2^{-1} \in C^\infty(a,b)$. Does this imply that $\phi_2 \circ \phi_1^{-1} \in C^\infty(c,d)$?

b. Given two homeomorphisms $X \xrightarrow[\sim]{\phi_1,\phi_2} Y$, suppose $\phi_1 \circ \phi_2^{-1} \in \mathcal{O}(X)$. Does this imply: $\phi_2 \circ \phi_1^{-1} \in \mathcal{O}(Y)$? [Question 7 of homework.0 is useful here.]

c. Prove: if a map $X \stackrel{f \in \mathcal{O}}{\to} Y$ is bijective, then it is analytically invertible.

d. Take a non-constant map $X \stackrel{f \in \mathcal{O}}{\to} Y$, with X-connected. Prove: for any $y \in Y$ the set of preimages, $f^{-1}(y)$ is discrete.