Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 3.

Submission date: 23.11.2025.

Questions to submit: 2.b. 3.a.i. 3.a.ii. 4.d. 6.b. 6.c.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. a. Take a surjection $X \stackrel{\pi}{\to} Y$, where X is a topological space. Take the quotient topology on Y. Prove: a function $Y \stackrel{f}{\to} (\dots)$ is continuous iff $f \circ \pi \in C^0(X)$.

b. Take the presentation $\mathbb{C}^2 \setminus o \stackrel{\pi}{\to} \mathbb{P}^1$. Verify: a function $\mathbb{P}^1 \supseteq \mathcal{U} \stackrel{f}{\to} \mathbb{C}$ is holomorphic iff $f \circ \pi \in \mathcal{O}(\pi^{-1}(\mathcal{U}))$. And similarly for meromorphic functions.

c. Do the same for $\mathbb{C} \stackrel{\pi}{\to} \mathbb{C}/_L$.

d. Go over all the details of the proof of "$\mathbb{P}^2$ is a compact connected $\mathcal{O}$-manifold".

2. a. Verify: any two distinct (complex) lines in $\mathbb{P}^2$ intersect in exactly one point. What has happenned to the parallel lines of $\mathbb{C}^2$?

b. Define the map $\mathbb{C}^2 \supseteq X = \{x \cdot y = 1\} \to \mathbb{C}^*$ by $(x, y) \to x$. Prove: this map extends to a biholomorphism $\mathbb{P}^2 \supset \bar{X} \stackrel{\sim}{\to} \mathbb{P}^1$. [Thus topologically $\bar{X} \approx S^2$]

c. Let $f(z, w)$ be a polynomial of degree d. Define $F(x_0, x_1, x_2) := x_0^d \cdot f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$. Verify:

- d is the minimal degree to ensure that F is a polynomial.
- d is the maximal degree to ensure that F is not divisible by x_0.

3. a. Consider the curve $X = \{y^2 = p(x)\} \subset \mathbb{C}^2$, where $p(x) \in \mathbb{C}[x]$.

i. Prove: X is smooth iff $p(x)$ has no multiple roots. When is $\bar{X} \subset \mathbb{P}^2$ smooth?

ii. Suppose X is smooth and consider the functions $f(x, y) = tan(\frac{y}{x})$, $g(x, y) = e^{\frac{y-1}{x}}$ on $\mathbb{C}^2$. At which points are the restrictions $f|_X, g|_X$ holomorphic? meromorphic?

b. A curve $\{f(\underline{z}) = 0\}$, in $\mathbb{C}^2$ or in $\mathbb{P}^2$, is called reducible if its defining equation factorizes non-trivially, i.e. $f = f_1 \cdot f_2$, where $f_i \neq const$. (Otherwise the curve is called irreducible.) How the (ir)reducibility of $X \subset \mathbb{C}^2$ is related to the (ir)reducibility of $\bar{X} \subset \mathbb{P}^2$?

c. Fix a projective algebraic plane curve $X \subset \mathbb{P}^2$. Prove: every projective line $L \subset \mathbb{P}^2$, has a non-empty intersection with X. Can we replace $\mathbb{P}^2$ by $\mathbb{C}^2$ here?

4. a. Verify: the notions of removable singularity/pole/essential singularity of a function do not depend on the choice of local coordinates.

b. For a Riemann surface $X = \{f(\underline{z}) = 0\} \subset \mathbb{P}^2$ fill all the details in the proof [from the class]: $M(X) \supseteq \{\frac{p(\underline{z})}{q(\underline{z})} |\ p, q - homogeneous,\ deg\ p = deg\ q,\ q|_X \neq 0\}$.

c. (Extension of targets) Prove: each $f \in M(X)$ defines a holomorphic map $X \stackrel{F}{\to} \mathbb{P}^1$. (Restriction of targets) State and prove the inverse statement.

d. Given a function $f \in M(X)$ and the corresponding map $X \stackrel{F}{\to} \mathbb{P}^1$, what is the relation between $mult_x(F)$ and $ord_x(f)$? (Distinguish between the zeros/poles of f and other points.)

e. Below X, Y are Riemann surfaces, with X is compact, connected. Suppose two maps $X \stackrel{f,g}{\to} Y$, coincide on infinite set of points. Prove: $f = g$.

5. Verify: the composition of holomorphic maps of Riemann surfaces is holomorphic. (This defines the map $Maps^{hol}(X, Y) \times Maps^{hol}(Y, Z) \to Maps^{hol}(X, Z)$.) In particular:

a. The composition of a holomorphic map and a holomorphic function is holomorphic.

b. The composition of a holomorphic map and a meromorphic function is meromorphic.

6. a. Verify: the multiplicity of a map at a point does not depend on the choice of local coordinates.

b. Take a holomorphic map $X_{compact} \stackrel{f}{\to} Y$. Prove: $mult_x(f) > 1$ only for a finite number of points.

c. Suppose $mult_x(f) = 1$. Prove that f is locally a biholomorphism at x. Conclude that f is a local bihilomorphism everywhere except for a finite set of points on X.