Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 4.

Submission date: 30.11.2025.

Questions to submit: 2.a. 2.b. 2.c. 3.b.iii. 5.a. 5.c. 6.b.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. Below $X \xrightarrow{f} Y$ is a non-constant analytic map of (connected) Riemann surfaces. Prove:

a. f is an open mapping.

b. If f is injective, then it is an isomorphism, $X \xrightarrow[\sim]{f} f(X) \subseteq Y$.

c. If X is compact then so is Y, and f is surjective.

d. For any $y \in Y$ the subset $\phi^{-1}(y) \subset X$ is discrete. If X is compact, then $\phi^{-1}(y)$ is finite.

2. a. Let $f(z) = \frac{4z^2(z^d-1)^2}{(2z-1)^2} \in M(\mathbb{C})$, consider the corresponding holomorphic map $\mathbb{P}^1 \xrightarrow{F} \mathbb{P}^1$. Describe its ramification data. What is $deg(F)$?

b. For a map $X \xrightarrow{f} Y$ and its presentation in (any) coordinates $f(z)$, verify: $mult_{z_o} f = \frac{(z-z_o)\cdot\partial_z f(z)}{f(z)}|_{z=z_o}$.

c. Take a smooth curve $X = \{f(x,y) = 0\} \subset \mathbb{C}^2$ and the projection $X \xrightarrow{\pi} \mathbb{C}^1_x$. Verify: $p \in X$ is a ramification point of X iff the tangent line $T_{(X,p)}$ is parallel to $\hat{y}$-axis iff $\partial_y f|_p = 0$.
Do this also for the projective case in homogeneous coordinates, $X = \{f(z_0, z_1, z_2) = 0\} \subset \mathbb{P}^2$.

3. a. Verify: a map of $deg(f) = 1$ has no ramifications and is a global isomorphism of Riemann surfaces.

b. Below X is a compact Riemann surface and $const \neq f \in M(X)$. Prove:

i. f has at least one zero and at least one pole.

ii. If f has precisely one pole (or one zero) then it is an isomorphism $X \xrightarrow[\sim]{f} \mathbb{P}^1$.

iii. $\sum\limits_{x\in X} ord_x(f) = 0$. (The number of zeros equals the number of poles.)

c. Project the curve $\{xy = 1\} \subset \mathbb{C}^2$ onto $\hat{y}$-axis. Is the degree of this projection constant?

4. a. Prove: any holomorphic map $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$ is presentable as $[z_0 : z_1] \to [p(z_0, z_1) : q(z_0, z_1)]$, for some homogeneous polynomials of the same degree d. When the degree of this map is just d?

b. An invertible holomorphic map $f \circlearrowright X$ is called an automorphism. Verify: the set of all the automorphism of X form a group, $Aut(X)$.

c. (Möbius transformations) Prove: the action $GL(2, \mathbb{C}) \circlearrowright \mathbb{C}^2$ descends to a holomorphic action $GL(2, \mathbb{C}) \circlearrowright \mathbb{P}^1$. Namely, for each $A \in GL(2, \mathbb{C})$ the map $[z_0 : z_1] \to [z_0 : z_1]A$ is holomorphic.
Write this action in the coordinate chart $\mathcal{U}_{z_o\neq 0}$. Prove: this correspondence defines a surjective homomorphism of groups $GL(2, \mathbb{C}) \twoheadrightarrow Aut(\mathbb{P}^1)$. What is the kernel?

5. a. Compute the genus of the Fermat curve $\{x^d + y^d = z^d\} \subset \mathbb{P}^2$.

b. Verify: the action $GL(3, \mathbb{C}) \circlearrowright \mathbb{P}^2$ preserves the degrees and genera of smooth algebraic curves.

c. Prove: any smooth projective cubic $X_{deg=3} \subset \mathbb{P}^2$ can be brought by the action $GL(3, \mathbb{C}) \circlearrowright \mathbb{P}^2$ to the Weierstraß form, which in the affine coordinates is $\{y^2 = x^3 + ax + b\}$.

d. Compute the genus of a(ny) smooth projective plane cubic.

e. Let $X \xrightarrow{f} Y$ be a non-constant holomorphic map of compact Riemann surfaces. Prove:

i. $g(X) \geq g(Y)$. (E.g. any map $\mathbb{P}^1 \to X_{g\geq 1}$ must be constant.)

ii. If $g(X) = g(Y) = 1$ then f is necessarily unramified.

iii. If $g(X) = g(Y) = 2$ then f is necessarily an isomorphism.

iv. The sum of ramification indices of f is even. (The ramification index at a point x is $mult_x(f) - 1$.)

6. a. Why do not we study “meromorphic maps of compact Riemann surfaces”?

b. Let $X = \{y = e^x\} \subset \mathbb{C}^2$. Describe $\bar{X} \subset \mathbb{P}^2$. (What is $\bar{X} \setminus X$?)