# Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

## Homework 5.

Submission date: 7.12.2025.
Questions to submit: 1.b. 2.f. 3.b. 4.a. 4.b.ii. 4.b.iii. 5.a.
(Either typed or in readable handwriting and scanned in readable resolution.)

**1.**a. Let $X \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth projective curve of degree $d$. Fix a point $p \notin X$ and a projective line $L \subset \mathbb{P}^2$ that does not pass through $p$. Define the map $X \xrightarrow{\pi} L \approx \mathbb{P}^1$ by projecting from $p$ to the line. Prove: $\pi$ is an analytic map of Riemann surfaces. Compute the degree of this map.

b. Repeat 1.a for the case: $p \in X$. What is $\pi(p)$ in this case? Is $\pi$ still analytic? What is the degree?

c. Take the map $\mathbb{C} \to \mathbb{C}^d$, $x \to (x, x^2, \dots, x^d)$. Verify: its image, $X$, is a Riemann surface. Take the projection onto $j$'th axis, $X \xrightarrow{\pi_j} \mathbb{C}_{x_j}$. Verify: $\pi_j$ is an analytic map. Find its degree and find the full ramification data.

**2.**a. Below $\mathbb{k} = \mathbb{R}, \mathbb{C}$. Take the projective line, $\mathbb{P}^1_{\mathbb{k}} := {}^{\mathbb{k}^2 \setminus o}/_{\mathbb{k}^*}$, and the projective plane, $\mathbb{P}^2_{\mathbb{k}} := {}^{\mathbb{k}^3 \setminus o}/_{\mathbb{k}^*}$. Identify the topological type of $\mathbb{P}^1_{\mathbb{R}}$ and $\mathbb{P}^2_{\mathbb{R}}$. Verify: $\mathbb{P}^2_{\mathbb{R}}$ is ($C^0$) presentable as $[aa]$.

b. Present the Klein bottle as a quotient: i. of the Möbius strip, ii. of the cylinder, iii. as ${}^{S^1 \times S^1}/_{(z,w) \sim (-z, \bar{w})}$, where $S^1 \times S^1 = \{(z, w),\ |z| = 1 = |w|\} \subset \mathbb{C}^1 \times \mathbb{C}^1$

c. Present $\mathbb{P}^2_{\mathbb{R}}$ as: i. a quotient of the Möbius strip, ii. a quotient of $\overline{Ball} \subset \mathbb{R}^2$, iii. as the Möbius strip with a disc glued to the boundary.

d. Present the *handle* ($S^1 \times S^1 \setminus Ball$) as a quotient of $[0, 1]^2$. Using this present $\sharp^g(S^1 \times S^1)$ as a quotient of a planar polygon.

e. Present the *crosscap* ($\mathbb{P}^2_{\mathbb{R}} \setminus Ball$) as a quotient of $[0, 1]^2$. Present $\sharp^g(S^1 \times S^1)\ \sharp^g(\mathbb{P}^2_{\mathbb{R}})$ as a quotient of a planar polygon.

f. Verify (either as it was done in the class or in other ways):

i. $S^2 \sharp X \overset{C^0}{\approx} X$. ii. $\mathbb{P}^2_{\mathbb{R}} \sharp \mathbb{P}^2_{\mathbb{R}} \approx Klein$. iii. $\mathbb{P}^2_{\mathbb{R}} \sharp Klein \approx \mathbb{P}^2_{\mathbb{R}} \sharp (S^1 \times S^1)$.

**3.**a. Take compact real surfaces $X_1, X_2$ (not necessarily orientable). Express $\chi(X_1 \sharp X_2)$ via $\chi(X_1), \chi(X_2)$.

b. Prove: $[a_1 b_1 a_1^{-1} b_1^{-1} \dots a_1 b_1 a_1^{-1} b_1^{-1}]$ is the shortest presentation of $\sharp^g(S^1 \times S^1)$.

**4.** In the class we saw how to "plug the holes" in a punctured Riemann surface.

a. A curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$ is said to "have a node" at $(0, 0)$ if $f(0, 0) = 0 = \partial_x f|_{(0,0)} = \partial_y f|_{(0,0)}$ and the Hessian matrix of $f$ at $(0, 0)$ is non-degenerate. Prove: by a local analytic change of coordinates at $(0, 0)$ we can bring $f$ to the form $x^2 + y^2$. A possible way:

- Use $GL(2, \mathbb{C})$ to bring $f$ to the form $x^2 + y^2 +$(terms of order$\geq 3$).
- Eliminate the higher order terms.

b. In the cases below start from $X \subset \mathbb{C}^2$, compactify it and eliminate the singularities by $\tilde{X} \to X$. Verify the (non-)connectedness of $\tilde{X}$.

i. $X = \{y^2 = x^3 + x^2\} \subset \mathbb{C}^2$.

ii. $X = \{y^p = x^q\} \subset \mathbb{C}^2$, with $gcd(p, q) = 1$. Verify: $\mathbb{P}^1 \overset{\mathcal{O}}{\underset{\sim}{\to}} \tilde{X}$.

iii. $X = \{y^p = x^q\} \subset \mathbb{C}^2$, with $gcd(p, q) = r$. Verify: $\amalg^r \mathbb{P}^1 \overset{\mathcal{O}}{\underset{\sim}{\to}} \tilde{X}$.

**5.**a. Let $X$ be a Riemann surface with punctures, and $\tilde{X}$ its smooth compactification. [E.g. obtained by elimination of singular points.] Prove: this compactification is unique. Namely, $\tilde{X}_1 \underset{\sim}{\to} \tilde{X}_2$ for any two compact Riemann surfaces $\tilde{X}_1 \supset X \subset \tilde{X}_2$, such that the sets $\tilde{X}_1 \setminus X$ and $\tilde{X}_2 \setminus X$ are discrete.

b. Let $X \subset \mathbb{P}^2_{\mathbb{C}}$ be a singular algebraic curve and let $\tilde{X}$ be the Riemann surface obtained by "eliminating" singular points of $X$. Take the natural projection $\tilde{X} \xrightarrow{\pi} X$. Fix an open $\mathcal{U} \subseteq \mathbb{C}^2$. Prove: any (holomorphic/meromorphic) function $f \in \mathcal{O}(\mathcal{U}), M(\mathcal{U})$ induces a holomorphic map $\pi^{-1}(\mathcal{U}) \xrightarrow{f|_X \circ \pi} \mathbb{P}^1$.