Introduction to Riemann Surfaces and Algebraic Curves

201.2.5101 Fall 2025 (Dmitry Kerner)

Homework 6.

Submission date: 15.12.2025.

Questions to submit: 2.a. 2.b. 3.b. 3.c. 3.g. 5.b. 5.c.

(Either typed or in readable handwriting and scanned in readable resolution.)

1. Take a (possibly singular) algebraic curve $X \subset \mathbb{P}^2$. Eliminate the singularities, $\tilde{X} \twoheadrightarrow X$. Suppose an open dense subset $\mathcal{U} \subset X$ admits a parametrization $\mathbb{C} \setminus (\text{finite set}) \xrightarrow{\sim} \mathcal{U}$. Prove: this parametrization extends to $\mathbb{P}^1 \xrightarrow{\sim} \tilde{X}$.

2. Take the curve-germ $(X, o) = \{y^p = h(x, y)\} \subset (\mathbb{C}^2, o)$, where $h(x, y) \in \mathbb{C}[x, y]$, $ord_o(h) = q \geq p \geq 2$. Suppose $h(x, y)$ contains the monomial $1 \cdot x^q$.

a. Suppose $gcd(p, q) = 1$. Prove: $(\mathbb{C} \setminus o, o) \overset{\mathcal{O}}{\xrightarrow{\sim}} (X \setminus o, o)$.

[Look for the parametrization of the form $x(t) = t^p$, $y(t) = t^q \cdot (1 + \Delta)$, where Δ is an unknown. Show that there exists the (unique) solution $\Delta(t) \in \mathbb{C}\{t\}$.]

b. Extend this to the case $gcd(p, q) = r \in \mathbb{N}$.

3. (Cyclic coverings of $\mathbb{P}^1$. An extension of "Hyperelliptic curves")

Let $X = \{y^p = h(x)\} \subset \mathbb{C}^2$ where $h(x) \in \mathbb{C}[x]$ of $deg(h) = d$, with no multiple roots. Take its closure $\bar{X} \subset \mathbb{P}^2$ and eliminate the singularities, $\tilde{X} \twoheadrightarrow \bar{X}$.

a. Find the singular points of $\bar{X}$. For each $x_o \in \bar{X}$ detect the $\mathcal{O}$-type of the germ $(X \setminus x_o, x_o)$, as in q.2.

b. Compute $g(\tilde{X})$.

c. Detect the topological types of $\tilde{X}, X, \bar{X}$.

d. The projection $X \xrightarrow{\pi} \mathbb{C}^1_x$ induces the morphism $\mathbb{C}(x) \xrightarrow{\pi^\sharp} M(\tilde{X})$. Verify: this is an embedding.

e. Define the automorphism $\sigma \circlearrowright X$ by $(x, y) \to (x, e^{\frac{2\pi i}{p}} y)$. Verify: it extends to an (analytic) automorphism $\sigma \circlearrowright \tilde{X}$.

f. Verify: a function $f \in M(\tilde{X})$ is $\sigma^\sharp$ invariant iff $f \in \pi^\sharp(\mathbb{C}(x))$.

g. Prove: every $f \in M(\tilde{X})$ is presentable as $s_0(x) + y \cdot s_1(x) + \cdots + y^{p-1} \cdot s_{p-1}(x)$, for $s_i(x) \in \mathbb{C}(x)$.

4.a. Take a Riemann surface X (not necessarily compact). Verify: $M(X)$ is a field.

b. For any morphism $X \overset{\phi \neq const}{\to} Y$ we get $M(X) \overset{\phi^\sharp}{\leftarrow} M(Y)$. What happens for $\phi = const$?

Is the map $M(\mathbb{P}^1) \xrightarrow{\phi^\sharp} M(\mathbb{C}^1)$ induced by the embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1$ an isomorphism?

5.a. Let $X = \{f(x, y) = 0\} \subset \mathbb{C}^2$, where $f(x, y) \in \mathbb{C}[x, y]$ is square-free.

i. Prove: a vector field $\xi_{\mathbb{C}^2}$ restricts to a vector field on X iff $\xi(f)$ is divisible by f.

[You can use Hilbert's Nullstellensatz]

ii. Verify: the vector field $\partial_x f \cdot \partial_y - \partial_y f \cdot \partial_x$ is tangent to all the curves $\{f(x, y) = c\}$ for $c \in \mathbb{C}$.

iii. Suppose f is weighted-homogeneous, i.e. $f(t^{\omega_x} x, t^{\omega_y} y) = t^p \cdot f(x, y)$, for some weights $\omega_x, \omega_y \in \mathbb{N}$. Verify: the Euler vector field $\omega_x x \cdot \partial_x + \omega_y y \cdot \partial_y$ restricts to a vector field on X.

b. For any $\mathcal{O}$-manifold X prove: vector fields acts on functions as differential operators. Namely, $\xi(f) \in \mathcal{O}(X), M(X)$ for any analytic vector field and any $f \in \mathcal{O}(X), M(X)$, and this action does not depend on the choice of local coordinates, and is $\mathbb{C}$-linear, and satisfies Leibniz rule.

c. Take a map of Riemann surfaces $X \xrightarrow{\phi} Y$. (Dis)Prove: i. If ϕ is injective then ϕ_* is injective.

ii. If ϕ is surjective then ϕ_* is surjective. iii. $\phi^*((\phi_* \xi)(f)) = \xi(\phi^*(f))$ for each $f \in \mathcal{O}(Y)$.

d. Define the commutator of two vector fields, $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$, via its action on functions, as in b. Prove $[\xi, \eta]$ is a vector field on X. Namely, this is an analytic differential operator of order one, independent of coordinate choices.

e. Fix vector fields ξ, η on $\mathbb{C}^N$ that are tangent to a submanifold $X \subset \mathbb{C}^N$. Verify: $[\xi, \eta]$ is tangent to X.